Math 116. Complex Analysis.
Final Exam - Practice Questions

Note : These problems are only intended to help you study. Questions based on any material covered in class may appear on the actual test.

1. Evaluate the integral:

$$\int_{\mathcal{C}} f(z)\, dz$$

when:

   (a) $f(z) = \frac{z+2}{z}$ and $\mathcal{C}$ is the semi-circle $\gamma(t) = 2e^{it}$, $0 \leq t \leq \pi$.

   (b) $f(z) = \pi e^{\pi \overline{z}}$ and $\mathcal{C}$ is the boundary of the square with vertices at the points 0, 1, $1+i$ and i, with counter-clock-wise direction.

2. Show that if $\mathcal{C}$ is the boundary of a triangle with vertices at the points 0, $3i$ and -4, with counter-clock-wise orientation, then:

$$\left| \int_{\mathcal{C}} (e^z - \overline{z})\, dz \right| \leq 60.$$

3. Without evaluating the integral, show that:

$$\left| \int_{\mathcal{C}} \frac{dz}{z^2 - 1} \right| \leq \frac{\pi}{3}$$

where $\mathcal{C}$ is the arc of the circle $|z| = 2$ from 2 to $2i$ that lies in the first quadrant.

4. Use Cauchy's formula to evaluate the following integrals:

   (a) $\int_{\mathcal{C}} \frac{e^{-z}}{z - \frac{\pi i}{2}}\, dz$

   (b) $\int_{\mathcal{C}} \frac{\cos z}{z(z^2+8)}\, dz$

   (c) $\int_{\mathcal{C}} \frac{z}{2z+1}\, dz$.

   In each case, $\mathcal{C}$ is the boundary of the square with vertices in $2+2i$, $-2+2i$, $-2-2i$ and $2-2i$, oriented with the counter-clock-wise direction.

5. Prove that an analytic function $f\colon \mathbb{C} \to \mathbb{C}$ satisfying $|f(z)| \leq 4$ for any $z \in \mathbb{C}$ must be constant.

6. Prove the fundamental theorem of algebra (any polynomial of degree greater or equal to one, with complex coefficients, has at least one zero).

7. Find the poles of the functions:

   (a) $f(z) = \frac{\sin(z)}{z(2z-\pi)}$

   (b) $f(z) = \frac{z^2-3z+2}{(z-1)^2(z-3)^3}$

   (c) $f(z) = \frac{z+1}{z^2+9}$.

   For each pole found, indicate its order and residue.

8. Find the residue at $z = 0$ of the functions:

   (a) $f(z) = \frac{1}{z+z^2}$.

   (c) $f(z) = \frac{z-\sin(z)}{z}$.

9. Show that the isolated singularities of the function

$$f(z) = \frac{z}{z^4+4}$$

   are poles. Determine the order of each pole and find the corresponding residues.

10. Use Residue's Theorem to evaluate the integral of each of these functions around the circle $|z| = 3$, in the counter-clock-wise direction.

    (a) $f(z) = \frac{e^{-z}}{z^2}$

    (b) $f(z) = \frac{z+1}{z^2-2z}$

    (c) $f(z) = \frac{5z-2}{z^2-z}$.

11. Use residues to evaluate the following improper integrals:

    (a) $\int_0^\infty \frac{1}{(x^2+1)^2}\, dx$

    (b) $\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\, dx$

    (c) $\int_0^\infty \frac{x\sin(2x)}{x^4+2}\, dx$

12. Determine the number of zeros (counting multiplicities) of the polynomial z^7-4z^3+z-1 which are situated inside the circle $|z| = 1$.