

Pg. 28, 1:

If

$$f(x, y) = u(x, y) + iv(x, y) \quad (1)$$

and

$$g(r, s) = \alpha(r, s) + i\beta(r, s), \quad (2)$$

then

$$g(f(x, y)) = \alpha(u(x, y), v(x, y)) + i\beta(u(x, y), v(x, y)). \quad (3)$$

Using the chain rule, we have

$$(\operatorname{Re} g(f))_x = \alpha_r(u, v)u_x + \alpha_s(u, v)v_x \quad (4)$$

$$(\operatorname{Re} g(f))_y = \alpha_r(u, v)u_y + \alpha_s(u, v)v_y \quad (5)$$

$$(\operatorname{Im} g(f))_x = \beta_r(u, v)u_x + \beta_s(u, v)v_x \quad (6)$$

$$(\operatorname{Im} g(f))_y = \beta_r(u, v)u_y + \beta_s(u, v)v_y. \quad (7)$$

Substituting the Cauchy-Riemann equations for f and g ,

$$u_x = v_y \quad (8)$$

$$u_y = -v_x \quad (9)$$

$$\alpha_r = \beta_s \quad (10)$$

$$\alpha_s = -\beta_r, \quad (11)$$

it is straightforward to see that $\operatorname{Re} g(f)$ and $\operatorname{Im} g(f)$ also satisfy the Cauchy-Riemann equations.

Pg. 28, 2:

Let $f(z) = z^2$. We'd like to show that f satisfies the Cauchy-Riemann equations. Set $z = x + iy$.

$$(x + iy)^2 = x^2 - y^2 + 2xyi. \quad (12)$$

$$(\operatorname{Re} f)_x = 2x \quad (13)$$

$$= (\operatorname{Im} f)_y \quad (14)$$

$$(\operatorname{Re} f)_y = -2y \quad (15)$$

$$= -(\operatorname{Im} f)_x. \quad (16)$$

The z^3 case is similar. Just multiply it out and check partials with respect to x and y .

Let f be an analytic function such that $|f|^2 = C$. We'd like to show that f is constant. This is obvious if $C = 0$, so let's also assume that $C > 0$. Set $f = u + iv$. Then

$$u^2 + v^2 = C. \tag{17}$$

Differentiating, we obtain

$$2uu_x + 2vv_x = 0 \tag{18}$$

$$2uu_y + 2vv_y = 0. \tag{19}$$

f is analytic, so we can use the Cauchy-Riemann equations to see that

$$2uu_x - 2vu_y = 0 \tag{20}$$

$$2uu_y + 2vu_x = 0 \tag{21}$$

$$2uv_y + 2vv_x = 0 \tag{22}$$

$$-2uv_x + 2vv_y = 0. \tag{23}$$

Setting $w_1 = (2u, -2v)$ and $w_2 = (2v, 2u)$, these four equations give

$$\nabla u \cdot w_1 = 0 \tag{24}$$

$$\nabla u \cdot w_2 = 0 \tag{25}$$

$$\nabla v \cdot w_2 = 0 \tag{26}$$

$$\nabla v \cdot w_1 = 0. \tag{27}$$

However, $C > 0$, so at each point z the vectors w_1 and w_2 are nonzero. They are also orthogonal and therefore a basis of \mathbf{R}^2 . This allows us to deduce from the previous equations that

$$\nabla u = (0, 0) \tag{28}$$

$$\nabla v = (0, 0), \tag{29}$$

and we conclude that f is a constant function.