Homework 7
Due Thursday night, November 9 (technically 2am Nov. 10)

Question 1. Recall that in class we used the free resolution from HW4 Q4(g) to compute for $G = \mathbb{Z}/2 = \{ 1, s \}$ that

$$H^k(\mathbb{Z}/2; M) = \begin{cases} M^G & k = 0 \\ \{ m \in M | sm + m = 0 \} & k = 1, 3, 5, \ldots \\ \{ m \in M | sm = m \} & k = 2, 4, 6, \ldots \end{cases}$$

For $G = \mathbb{Z}/n = \{ 1, s, \ldots, s^{n-1} \}$, find a similar description of $H^k(\mathbb{Z}/n; M)$ for a $\mathbb{Z}G$-module $M$.

(Hint: find a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module; note that $\mathbb{Z}G \cong \mathbb{Z}[s]/(s^n - 1)$.)

The resolution will again be 2-periodic just like for $\mathbb{Z}[s]/(s^2 - 1)$.

Question 2. Let $G$ be a group.

(a) Prove that $H^0(G; \mathbb{Z}G) \cong \mathbb{Z}$ if $G$ is finite, and $H^0(G; \mathbb{Z}G) = 0$ if $G$ is infinite.

(b) Prove that $H^1(G; \mathbb{Z}G) \neq 0$ if $G = \mathbb{Z} = \{ \ldots, t^{-1}, 1, t, \ldots \}$.

(c) (Hard, very optional) Can you find another group for which $H^1(G; \mathbb{Z}G) \neq 0$?

Question 3. Let $L/K$ be a finite Galois extension with Galois group $G = \text{Gal}(L/K)$. The unit group $L^\times$ is an abelian group with an action of $G$, so we may consider the group cohomology $H^k(G; L^\times)$. A theorem of Noether states that $H^1(G; L^\times) = 0$; you may assume this without proof.

(a) Use Noether’s theorem to prove that if $\text{Gal}(L/K)$ is generated by a single element $s$, then every element $\ell \in L$ with norm 1 has the form $s(z)/z$ for some $z \in L$.

(b) Use part (a) to give a parametrization in two rational parameters of the rational points on the unit circle:

$$S^1(\mathbb{Q}) = \{ (x \in \mathbb{Q}, y \in \mathbb{Q}) | x^2 + y^2 = 1 \}.$$

That is, give two rational functions $x(a, b) \in \mathbb{Q}(a, b)$ and $y(a, b) \in \mathbb{Q}(a, b)$ such that the resulting function $f : \mathbb{Q}^2 \to \mathbb{Q}^2$ given by $(a, b) \mapsto (x(a, b), y(a, b))$ has image $S^1(\mathbb{Q})$.

NOTE: $f$ might not not actually be a function $\mathbb{Q}^2 \to \mathbb{Q}^2$ because a rational function like $x(a, b)$ might take the value $\infty$ sometimes. So formally this should say “the resulting function $f : \{ \text{subset of } \mathbb{Q}^2 \text{ where both } x \text{ and } y \text{ are not } \infty \} \to \mathbb{Q}^2$ has image $S^1(\mathbb{Q})$.”

(cont.)
Given a chain complex $C = \cdots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow 0$ and a chain map $f: C \rightarrow C$:

We call $f$ an \textit{involution} if $f \circ f = \text{id}$.

We call $f$ a \textit{weak involution} if there is a \textit{homotopy} $f \circ f \sim \text{id}$.

**Question 4.** Give an example of a chain complex $C$ and a weak involution $f: C \rightarrow C$ that is not an involution.

**Question 5.** (Optional, replaces Q4) Give an example of a chain complex $C$ and a weak involution $f: C \rightarrow C$ that is not homotopic to an involution.

(That is, there does not exist any involution $g: C \rightarrow C$ with $g \circ g = \text{id}$ and $f \sim g$.)

**Question 6.** (Stupid hard, worth 0 points; only respect and admiration)
OK to discuss with classmates and even turn in joint solution, but \textit{not} to discuss with others outside class. Can be turned in any time before the last day of class.

Give an example of a chain complex $C$ and a weak involution $f: C \rightarrow C$ that cannot be made homotopic to an involution even if we replace $C$ by a homotopy equivalent complex.

More precisely, note that given a homotopy equivalence $\varphi: C \rightarrow D$ with homotopy inverse $\psi: D \rightarrow C$, then the map $\varphi \circ f \circ \psi: D \rightarrow D$ will be a weak involution.

Say that $f$ is “fixable” if for some such $\varphi$ and $\psi$ this weak involution $\varphi \circ f \circ \psi$ is homotopic to an involution $g: D \rightarrow D$.

Give an example of a chain complex $C$ and weak involution $f: C \rightarrow C$ that is not “fixable”.