Math 210A: Modern Algebra  
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Homework 2  
Due Thursday night, October 5 (technically 5am Oct. 6)

Note that Q3 says “optional, replaces Q2”. This means that this question is optional (really!); however, if you write up and submit this optional question Q3 you do not have to submit Q2.

Recall that:
- a domain is a commutative ring in which \(xy = 0 \implies (x = 0 \text{ or } y = 0)\);
- a prime ideal \(P \subseteq R\) is an ideal for which \(xy \in P \implies (x \in P \text{ or } y \in P)\) 
  (or equivalently, for which the quotient \(R/P\) is a domain); and
- a maximal ideal \(m \subseteq R\) is one for which the quotient ring \(R/m\) is a field.

**Question 1.** Let \(R\) be a commutative ring, and let \(I = \{r \in R \mid \exists k > 0 \text{ such that } r^k = 0\}\).

(a) Prove \(I\) is an ideal.

(b) Prove \(I\) is the intersection of all the prime ideals of \(R\).
   You may use without proof the following fact, a consequence of Zorn’s lemma: if \(S\) is a subset of \(R\) satisfying \(0 \notin S\) and \(S \cdot S \subseteq S\), then the set of ideals \(J \subseteq R\) for which \(J \cap S = \emptyset\) has a maximal element.

**Question 2.** Let \(R = C^0([0,1])\) be the ring of real-valued continuous functions on the closed interval \([0,1]\). For every point \(p \in [0,1]\), we obtain a maximal ideal \(m_p = \{f \in R \mid f(p) = 0\}\).

Prove that every maximal ideal of \(R\) is of the form \(m_p\) for a unique \(p \in [0,1]\).

(Hint: You may wish to recall that \([0,1]\) is compact, which means that for any collection of open intervals covering it, there is some finite subcollection that still covers it.)

Note that this means that you can recover the set \([0,1]\) just from the ring \(R\).
(This actually works for any compact Hausdorff space, not just \([0,1]\); the proof is the same.)

(Optional, to think about: can you also recover the topology on \([0,1]\) from the ring \(R\)?)
Question 3 (optional, replaces Q2). [This question is very hard, 100% optional, and cannot be done without material from outside this course.]

Let \( R = C^\infty(S^1; \mathbb{C}) \) be the ring of complex-valued smooth functions on the circle \( S^1 \), which for concreteness I will realize as smooth 1-periodic functions on \( \mathbb{R} \):

\[
R \cong \{ f \in C^\infty(\mathbb{R}; \mathbb{C}) \mid f(x + 1) = f(x) \}.
\]

The proof of Q2 applies in exactly the same way to \( R \), showing that every maximal ideal of \( R \) is of the form \( \mathfrak{m}_p = \{ f \in R \mid f(p) = 0 \} \) for a unique \( p \in [0, 1) \approx S^1 \); you do not have to prove this.

(The complex-valued vs real-valued is not an important point, it just simplifies the following.)

For any \( f \in R \) we can define complex numbers \( a_n \in \mathbb{C} \) for all \( n \in \mathbb{Z} \) by

\[
a_n = \int_0^1 f(x)e^{-2\pi inx} \, dx.
\]

(Remark: It is a fact that these numbers decay rapidly as \( n \to \infty \), in the sense that for all \( k \geq 0 \) we have \( n^k |a_n| \to 0 \) and \( n^k |a_{-n}| \to 0 \) as \( n \to +\infty \).)

Let \( S \subset R \) be the subring consisting of those functions for which \( a_{-1} = a_{-2} = \cdots = 0 \), i.e.

\[
S = \left\{ f \in R \left| \int_0^1 f(x)e^{-2\pi inx} \, dx = 0 \text{ for all } n < 0 \right. \right\}
\]

(You do not have to prove that \( S \) is a subring of \( R \), though you might benefit from thinking about why it is.) For every \( p \in [0, 1) \), we still have a maximal ideal \( \mathfrak{m}_p \subset S \) given by \( \mathfrak{m}_p = \{ f \in S \mid f(p) = 0 \} \).

Exhibit a maximal ideal of \( S \) that is not of this form, and ideally exhibit two such maximal ideals. (If you really want a challenge: can you classify all maximal ideals of \( S \)? Warning: I do not know that this is possible using things you know. But you could at least come up with a guess, even if you can’t completely prove it’s correct.)

Question 4. Let \( a \in \mathbb{Z} \) and \( b \in \mathbb{Z} \) be coprime. Let \( C \) be any abelian group, and let

\[
f : (\mathbb{Z}/a\mathbb{Z}) \times (\mathbb{Z}/b\mathbb{Z}) \to C
\]

be a \( \mathbb{Z} \)-bilinear map. Prove that \( f = 0 \).

Question 5. Let \( N \) be a submodule of the \( R \)-module \( M \). Prove that if \( N \) is finitely generated and \( M/N \) is finitely generated, then \( M \) is finitely generated.

Question 6. Let \( M \) be a finitely generated \( R \)-module. Let \( \pi : M \to R^n \) be a surjective homomorphism, and let \( K = \ker(\pi) \). Prove that the \( R \)-module \( K \) is finitely generated.

Localization of modules, which features in the next question, will be covered on Monday.

Question 7. Let \( R \) be a commutative ring, and let \( S \subset R \) be a multiplicative set \( (S \cdot S \subset S) \). Let \( M \) be a finitely generated \( R \)-module. Prove that the localization \( S^{-1}M \) satisfies

\[
S^{-1}M = 0 \iff \exists s \in S \text{ with } s \cdot M = 0.
\]

(cont)
Question 8. Let \( f: X \to Y \) be a homomorphism of \( R \)-modules.

(a) Consider all pairs \((A, \alpha)\) of an \( R \)-module \( A \) and a homomorphism \( \alpha: A \to X \) with \( f \circ \alpha = 0 \).

\[
\begin{array}{c}
A \xrightarrow{\alpha} X \\
\downarrow 0 \quad \downarrow f \\
0 \quad Y
\end{array}
\]

You will prove that there exists a “universal” such pair. Specifically, you must construct some \((M, \mu: M \to X)\) with \( f \circ \mu = 0 \) with the property that:

for any \((A, \alpha: A \to X)\) with \( f \circ \alpha = 0 \), there exists a unique \( a: A \to M \) such that \( \alpha = \mu \circ a \).

\[
\begin{array}{c}
A \xrightarrow{a} M \xrightarrow{\mu} X \\
\downarrow 0 \quad \downarrow 0 \\
0 \quad Y
\end{array}
\]

(We might abbreviate this property as saying roughly: “Every \( \alpha \) with \( f \circ \alpha = 0 \) factors uniquely through \( M \”).)

(b) On the other side, consider all pairs \((B, \beta: Y \to B)\) with \( \beta \circ f = 0 \).

\[
\begin{array}{c}
X \\
\downarrow f \\
Y \xrightarrow{\beta} B
\end{array}
\]

Prove that there exists a “universal” such pair, by constructing some \((N, \nu: Y \to N)\) with \( \nu \circ f = 0 \) with the property that:

for any \((B, \beta: Y \to B)\) with \( \beta \circ f = 0 \), there exists a unique \( b: N \to B \) such that \( \beta = b \circ \nu \).

\[
\begin{array}{c}
X \\
\downarrow f \\
Y \xrightarrow{\nu} N \xrightarrow{b} B \\
\downarrow 0 \quad \downarrow \beta
\end{array}
\]

(We might abbreviate this property as saying roughly: “Every \( \beta \) with \( \beta \circ f = 0 \) factors uniquely through \( N \”).)