Math 196-47, Mr. Church

Some sample problems for the final

(covering material since the second midterm).

I think everyone is fairly comfortable with the material that was on the first midterm. For material we covered before the second midterm, the sheet I made up back then still applies. Remember that orthogonal matrices and orthogonal projections are covered in Chapters 9.1 and 9.2. For material from Chapter 8.3, see "Homework" 14 on Chalk.

- Given an explicit matrix, find the characteristic polynomial; find the eigenvalues; find the algebraic and geometric multiplicities; find a basis for the eigenspaces; determine whether or not it is diagonalizable.
- Write down a matrix so that $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ are eigenvectors (with whatever eigenvalues you want).
- Exercise 7.1.10: Let A be an arbitrary $n \times n$ matrix with the special property that the sum of the entries of each row is 7. Show that 7 is an eigenvalue of A. (Hint: find an eigenvector.)
- Exercise 7.2.3.
- Exercise 7.2.10.
- Exercise 7.3.1. (We did not cover Section 7.3, but this particular exercise doesn't use anything from the section.)
- (1) Given $T\colon \mathbb{R}^n \to \mathbb{R}^m$, show that if $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)$ is linearly independent, then $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$.
  (2) Is the converse true? That is, if $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis, does it follow that $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is linearly independent?
  (3) Is (1) true or false if you replace "is linearly independent" with "spans $\mathbb{R}^m$"? How about (2)?
- Define the "zero transformation" $Z\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $Z(\vec{v}) = \vec{0}$ for all $\vec{v}$. Assume that $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $T \circ T = Z$; is it necessarily true that $T = Z$? Explain or give a counterexample.

The following question is harder, but still may be good practice.

- Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies $T \circ T = T$. Show that every vector $\vec{v} \in \mathbb{R}^n$ can be written uniquely as
  $$\vec{v} = \vec{v}^{\mathrm{k}} + \vec{v}^{\mathrm{i}},$$
  where $\vec{v}^{\mathrm{k}} \in \ker T$ and $\vec{v}^{\mathrm{i}} \in \operatorname{im} T$.