Math 196-47, Mr. Church

Notes from class, Friday, May 8, 2009.

Orthogonal complements

- First, recall that $\langle v, w\rangle = ||v||\ ||w|| \cos\theta$, where θ is the angle between v and w. This can be used to calculate the angle between two vectors.
- Since lines are perpendicular if they meet at a right angle (so that $\cos\theta = 0$), we make the following definition:

  Definition. We say that v and w are perpendicular (or orthogonal) and write $v \perp w$ if $\langle v, w\rangle = 0$.

- Given a vector $v \in \mathbb{R}^n$, we define

  $$v^\perp = \left\{ w \in \mathbb{R}^n \middle| \langle v, w\rangle = 0 \right\}$$

  to be the set of vectors orthogonal to v. More generally, if S is any collection of vectors in $\mathbb{R}^n$, we define

  $$S^\perp = \left\{ w \in \mathbb{R}^n \middle| \langle v, w\rangle = 0 \text{ for all vectors } v \in S \right\}$$

  We call $v^\perp$ or $S^\perp$ the "orthogonal complement" to v or S respectively.

  Theorem. *$S^\perp$ is always a subspace. (In particular, $v^\perp$ is a subspace.)*

  Remark. The proof of this theorem uses the "bilinearity" of the inner product $\langle\cdot,\cdot\rangle$.

Orthogonal matrices

- Now we change focus somewhat. Given an $n \times n$ matrix A, we can think of a "function" that takes in a vector $v \in \mathbb{R}^n$ and spits out the vector Av. We can ask the following question: for which matrices A does this function preserve the length of all vectors? That is, for which A is it true that $||Av|| = ||v||$ for all vectors $v \in \mathbb{R}^n$? (Matrices of this kind will be called "orthogonal matrices" and we will continue with them on Monday.)
- The first observation is that a matrix that preserves all lengths also preserves all angles. More precisely:

  Theorem. *If $||Av|| = ||v||$ for all vectors $v \in \mathbb{R}^n$, then it is also true that $\langle Av, Aw\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{R}^n$.*

Remark. The proof uses the fact that

$$2\langle v, w\rangle = ||v+w||^2 - ||v||^2 - ||w||^2 .$$

- Now let's restrict our attention to $\mathbb{R}^2$ (that is, to the plane). First notice that a rotation of the plane by the angle θ preserves the length of all vectors. So does a reflection across a line through the origin. We will focus on the first kind, and we define

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Theorem. *R_θ rotates the plane by angle θ. That is, if you take the vector v and rotate it by angle θ, you get the vector $R_\theta v$.*

Proof. (This is just in case anyone wants to see how the proof works. I didn't do this in class.) If v in polar coordinates is (r, ϕ), then after rotating by angle theta it will be $(r, \phi + \theta)$. That is, if we write $v = \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix}$, we want to show that $R_\theta v = \begin{bmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{bmatrix}$. Writing out the matrix multiplication, we have that

$$R_\theta v = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix} = \begin{bmatrix} r(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ r(\sin\theta\cos\phi + \cos\theta\sin\phi) \end{bmatrix}$$

But the trigonometric addition formulas say that $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ and $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$, so we conclude that

$$R_\theta v = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix} = \begin{bmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{bmatrix}$$

as desired. □