Elementary Number Theory

Math 175, Section 30, Autumn 2010
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Homework 4

Due Tuesday, November 2 in class.[1]

Question 1. Let $P(x)$ be a polynomial with integer coefficients of the form

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0,$$

with $a_i \in \mathbb{Z}$ for each i.

a) Prove that if $r \in \mathbb{Q}$ is a root of $P(x)$ — meaning that $P(r) = 0$ — then:

   i) r is an integer, and

   ii) $r | a_0$.

b) If the degree n is odd, the constant term a_0 is odd, and $P(x)$ has an odd number of odd coefficients,

$$\text{i.e. the size of the set } \left\{1 \le i \le n-1 \;\middle|\; a_i \text{ is odd}\right\} \text{ is odd}$$

then $P(x)$ has at least one irrational real root.

(Some examples of polynomials satisfying this condition are $x^3 - 2x^2 + 9x + 1$ and $x^5 - 11x^4 + 7$.)

[1] Election day!