Elementary Number Theory

Math 175, Section 30, Autumn 2010
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Homework 2

Due Tuesday, October 19 in class.

In Homework 1 we defined the number system $\mathbb{Z}[i]$, consisting of formal expressions of the form $a + bi$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, and defined addition and multiplication.

Let $x, y \in \mathbb{Z}[i]$. Just as we did in Script 1, we say that y divides x (and that y is a *divisor* of x, and write $y|x$) if there is some $z \in \mathbb{Z}[i]$ such that $x = y \cdot z$.

Question 1. Prove that any nonzero element of $\mathbb{Z}[i]$ has finitely many divisors in $\mathbb{Z}[i]$.

Question 2. Let $x, y \in \mathbb{Z}[i]$. If $x|y$ and $y|x$, we say that y is an *associate* of x. Prove that if y is an associate of x, then y is either equal to x, $-x$, ix, or $-ix$.

Question 3 (The Division Algorithm for $\mathbb{Z}[i]$)**.** If $x, y \in \mathbb{Z}[i]$ and $y \neq 0$, then there exist $q \in \mathbb{Z}[i]$ and $r \in \mathbb{Z}[i]$ so that

$$x = yq + r \qquad \text{and} \qquad N(r) < N(y).$$

Question 4. Note that we did not require q and r to be unique in Question 3. Give an example of elements $x \in \mathbb{Z}[i]$ and $y \in \mathbb{Z}[i]$ for which there is more than one possible choice of q and r. Can you find an example where there are more than *two* possible choices of q and r?