# Math 120 – Spring 2018 – Prof. Church
# Midterm Exam: due 11:59pm on Monday, May 14

Please put your name on the *next page*, not this one.

Your exam **must** be submitted on Canvas by 11:59pm on Monday, November 13 or you will receive a zero. (Note Canvas marks submissions between 11h59m00s and 11h59m59s as late, but I will still accept them.)

This exam is open-book and open-notes, but closed-everything-else. (Needless to say, you should not discuss this exam with anyone.) In your proofs you may use any theorem from class, and any theorem or proposition from the chapters of Dummitt–Foote that we covered:

Chapter 1; Chapter 2 except 2.5; Chapter 3;
Chapter 4; Chapter 5.1, 5.4, 5.5; Chapter 6.3.

(We did not cover 2.5 or 4.6, but there's nothing there that helps with the exam so you don't have to avoid them.) You can use the statements of any question or exercise that was assigned as homework, but not any other exercises in the book (nor things labeled as "examples"). You can read the homework solutions if you like, but you cannot quote them as a reference. You do not have to give citations from the book for every result you use, but if you are at all unsure about the statement or why the result applies, it might be a good idea to look it up and make sure.

Questions? E-mail Prof. Church at `tfchurch@stanford.edu`.

There are 5 questions worth 100 points total on this exam.

| Total | 1 | 2 | 3a | 3b | 3c | 3d | 4 | 3a | 3b | 3c | 3d | 3e | 3f | 3g |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | |
| 100 points | 15 points | 15 points | 20 points | | | | 15 points | 35 points | | | | | | |

Name: ______________________________

Signature: ______________________________

**Question 1** (15 points)**.** Setup: Let $p$ be a prime number. Recall from §1.4 that $\mathrm{GL}_m(\mathbb{Z}/p\mathbb{Z})$ denotes the finite group of invertible $m \times m$ matrices over $\mathbb{Z}/p\mathbb{Z}$ under matrix multiplication:

$$\mathrm{GL}_m(\mathbb{Z}/p\mathbb{Z}) = \{\ m \times m \text{ matrices } A \text{ with entries in } \mathbb{Z}/p\mathbb{Z} \ \big|\ \det A \neq 0 \in \mathbb{Z}/p\mathbb{Z}\ \}$$

You may use without proof[1] that $|\mathrm{GL}_m(\mathbb{Z}/p\mathbb{Z})| = (p^m - 1)(p^m - p)(p^m - p^2)\cdots(p^m - p^{m-1})$.

Say that a matrix $A \in \mathrm{GL}_m(\mathbb{Z}/p\mathbb{Z})$ is[2] *upper-triangular* if $A$ has 1's on the diagonal and 0's below the diagonal, i.e. if $A$ has the form:

$$A = \begin{bmatrix} 1 & * & * & \cdots & * & * \\ 0 & 1 & * & \cdots & * & * \\ 0 & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & * \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Let $UT_m$ denote the subgroup consisting of upper-triangular matrices:

$$UT_m = \{A \in \mathrm{GL}_m(\mathbb{Z}/p\mathbb{Z}) \mid A \text{ is upper-triangular}\}$$

You may use without proof that $UT_m$ is a subgroup of $\mathrm{GL}_m(\mathbb{Z}/p\mathbb{Z})$.

Question: Suppose that $G$ is a finite group. Prove that the following are equivalent:

(A) $G$ is isomorphic to a subgroup of $UT_m$ for some $m$.

(B) the order of $G$ is a power of $p$: $|G| = p^k$.

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[1]For convenience, you are also welcome to omit the bars over elements of $\mathbb{Z}/p\mathbb{Z}$.
(e.g. you can write $2 + 2 = 1 \in \mathbb{Z}/3\mathbb{Z}$ rather than $\overline{2} + \overline{2} = \overline{1}$)

[2]Some people prefer the term "strictly upper-triangular", but let's not worry about that.

For Questions 2, 3, and 4, say that two points in $\mathbb{Z}^2$ are *adjacent* if they differ by exactly 1 in exactly one coordinate. In other words, each point $(a, b) \in \mathbb{Z}^2$ is adjacent to exactly four points, namely:

$$\begin{array}{ccccc} & & (a, b+1) & & \\ & & | & & \\ (a-1, b) & — & (a, b) & — & (a+1, b) \\ & & | & & \\ & & (a, b-1) & & \end{array}$$

Say that a bijection $f\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ is *adjacency-preserving* (for short, that $f$ is an *ap-bijection*) if

$$p \text{ adjacent to } q \qquad \Longleftrightarrow \qquad f(p) \text{ adjacent to } f(q)$$

Let $G = APB(\mathbb{Z}^2)$ be the group of adjacency-preserving bijections $f\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ under composition; you may assume without proof that $G$ is a group.

**Question 2** (15 points).
Give a **precise description of all the conjugacy classes** in the group $G = APB(\mathbb{Z}^2)$.
Which conjugacy classes contain one element? Which conjugacy classes contain $n$ elements for $n \in \mathbb{N}$? Which (if any) contain infinitely many elements? Justify your answers.

Possible hint: start by figuring out a good way to list/label all the elements of $G$.

**Question 3** (20 points).
This question has 4 parts, each worth 5 points. For each of the following, either give an example of the normal subgroup $N_k$ or prove that no such normal subgroup exists. (When giving examples, as long as your example is correct, you do not have to *prove* that it is a normal subgroup; however it might be safer to sketch the proof, so you can still get partial credit if there's a mistake.)

(a) Does $G$ have a normal subgroup $N_2$ of index $[G : N] = 2$?

(b) Does $G$ have a normal subgroup $N_4$ of index $[G : N] = 4$?

(c) Does $G$ have a normal subgroup $N_5$ of index $[G : N] = 5$?

(d) Does $G$ have a normal subgroup $N_{72}$ of index $[G : N] = 72 = 8 \cdot 9 = 2^3 \cdot 3^2$?

**Question 4** (15 points).
Suppose that $G \twoheadrightarrow H$ is a surjective homomorphism, and $H$ is an abelian group.
Prove that $|H|$ is finite and $|H|$ divides 8.

**Question 5** (35 points)**.** This question has 7 parts, each worth 5 points. In your solution to one part, you **may assume the results of all preceding parts**, whether or not you solved those earlier parts. For example, in your solution to (f) you may assume that $|N| = 56$ since this is the result of (e), even if you have not solved (e).

Let $A$ be the abelian group $A = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = (\mathbb{Z}/2\mathbb{Z})^3$, which has order 8. Let $G = \mathrm{Aut}(A)$ be the group of automorphisms of $A$. Its order is $|G| = 168 = 2^3 \cdot 3 \cdot 7$.

(You may assume this without proof.)

In this problem, you will prove that $G$ is a *simple group*: it contains no normal subgroups other than $\{1\}$ and $G$ itself. Therefore assume that $N \triangleleft G$ is a normal subgroup of $G$, and that $\{1\} \neq N \neq G$. We will eventually prove that this leads to a contradiction.

(a) Let $v$ be the nonzero vector $v = (1, 0, 0) \in A$, and let $H = \mathrm{Stab}_G(v)$ be its stabilizer. Prove that $N$ is not contained in $H$.

(Hint: otherwise, show that $N$ would be contained in all the conjugates of $H$.)

(b) Prove that $N$ contains an element of order 7.

(c) Prove that there is more than one 7-Sylow subgroup of $G$.

(One possible route: show that $f(a, b, c) = (c, a + c, b)$ and $g(a, b, c) = (b, c, a + b)$ define two order-7 elements $f \in G$ and $g \in G$ that do not commute. There are other approaches as well.)

(d) Show that $G$ contains exactly 48 elements of order 7.

(e) Prove that all 48 elements of $G$ of order 7 are contained in $N$, and conclude that $|N| = 56$.

(f) Let $K$ be the subgroup of $G$ consisting of automorphisms that permute the three elements $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ of $A$. There is an isomorphism $\psi\colon S_3 \xrightarrow{\cong} K$.

(You may use this without proof.)

Prove that $K$ is contained in $N$.

(g) Obtain a contradiction from the preceding parts.

(a) Let $v$ be the nonzero vector $v = (1,0,0) \in A$, and let $H = \mathrm{Stab}_G(v)$ be its stabilizer. Prove that $N$ is not contained in $H$.

(Hint: otherwise, show that $N$ would be contained in all the conjugates of $H$.)

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| Recall: | $A = (\mathbb{Z}/2\mathbb{Z})^3$ | $G = \mathrm{Aut}(A)$ | $\lvert G\rvert = 168 = 2^3 \cdot 3 \cdot 7$ |
| | $v = (1,0,0)$ | $H = \mathrm{Stab}_G(v)$ | |
| | $N \triangleleft G$ | $\{1\} \neq N \neq G$ | |

(b) Prove that $N$ contains an element of order 7.

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| | $N \triangleleft G$ | $\{1\} \neq N \neq G$ | |
| | (a) $N$ is not contained in $H$ | | |

(c) Prove that there is more than one 7-Sylow subgroup of $G$.

(One possible route: show that $f(a,b,c) = (c, a+c, b)$ and $g(a,b,c) = (b, c, a+b)$ define two order-7 elements $f \in G$ and $g \in G$ that do not commute. There are other approaches as well.)

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| | (a) $N$ is not contained in $H$ | | |
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(d) Show that $G$ contains exactly 48 elements of order 7.

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(a) $N$ is not contained in $H$

(b) $N$ contains an element of order 7

(c) $n_7(G) > 1$

(e) Prove that all 48 elements of $G$ of order 7 are contained in $N$, and conclude that $|N| = 56$.

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| | $N \triangleleft G$ | $\{1\} \neq N \neq G$ | |
| | (a) $N$ is not contained in $H$ | | (d) $G$ contains 48 elements of order 7 |
| | (b) $N$ contains an element of order 7 | | |
| | (c) $n_7(G) > 1$ | | |

(f) Let $K$ be the subgroup of $G$ consisting of automorphisms that permute the three elements $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ of $A$. There is an isomorphism $\psi\colon S_3 \xrightarrow{\cong} K$.

(You may use this without proof.)

Prove that $K$ is contained in $N$.

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| | $N \triangleleft G$ | $\{1\} \neq N \neq G$ | |
| | (a) $N$ is not contained in $H$ | | (d) $G$ contains 48 elements of order 7 |
| | (b) $N$ contains an element of order 7 | | (e) $N$ contains 48 elements of order 7 |
| | (c) $n_7(G) > 1$ | | (e) $\lvert N\rvert = 56$ |

(g) Obtain a contradiction from the preceding parts.

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| Recall: | $A = (\mathbb{Z}/2\mathbb{Z})^3$ | $G = \mathrm{Aut}(A)$ | $\lvert G\rvert = 168 = 2^3 \cdot 3 \cdot 7$ |
| | $v = (1, 0, 0)$ | $H = \mathrm{Stab}_G(v)$ | |
| | $N \triangleleft G$ | $\{1\} \neq N \neq G$ | |
| | (a) $N$ is not contained in $H$ | | (d) $G$ contains 48 elements of order 7 |
| | (b) $N$ contains an element of order 7 | | (e) $N$ contains 48 elements of order 7 |
| | (c) $n_7(G) > 1$ | | (e) $\lvert N\rvert = 56$ |
| | | | (f) the subgroup $K \cong S_3$ is contained in $N$ |