Homework 9

Math 120 (Thomas Church, Spring 2018)

Due Wednesday, June 6 at 11:59pm.

Question 1. Write down a ring homomorphism (no proof required)

$$f \quad \text{from} \quad R = \mathbb{Z}[\sqrt{11}] = \left\{ a + b\sqrt{11} \,\middle|\, a, b \in \mathbb{Z} \right\} \quad \text{to} \quad S = \mathbb{Z}/35\mathbb{Z}.$$

$f\big(a + b\sqrt{11}\big) =$

Question 2. Let $R \subset \mathbb{R}[x]$ be the subring of $\mathbb{R}[x]$ consisting of polynomials whose coefficient of x is 0:

$$\begin{aligned} R &= \{ \ f(x) = a_0 \qquad\qquad + a_2x^2 + \cdots + a_nx^n \ \mid a_i \in \mathbb{R} \ \} \\ \mathbb{R}[x] &= \{ \ f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \ \mid a_i \in \mathbb{R} \ \} \end{aligned}$$

You proved on HW8 Q1 that R is not a PID. Is R a UFD[1]? Prove or disprove.

Question 3. Given a polynomial $p(x) \in R[x]$ and an element $a \in R$, we say that a is a *root* of $p(x)$ if $p(a) = 0 \in R$.

(For example, in HW6 Q6 you proved that $p(x) = x^2 - x$ has 4 roots in the ring of infinite-integers $\mathbb{Z}/10^\infty\mathbb{Z}$.)

Prove that if R is a domain and $p(x)$ has degree n, then $p(x)$ has at most n roots in R.

Question 4. Let $p(x) \in \mathbb{C}[x]$ be a nonzero polynomial. Consider the following two properties of $p(x)$:

(A) The quotient ring $\mathbb{C}[x]/(p(x))$ is isomorphic to a product ring $\mathbb{C} \times \cdots \times \mathbb{C}$.

(B) The polynomial $p(x)$ has no repeated roots.[2]

Prove that these two properties are equivalent: $(A) \iff (B)$.

[1] from lecture Wed May 30, or §8.3 if you can't wait

[2] The fundamental theorem of algebra says any nonzero polynomial $p(x) \in \mathbb{C}[x]$ factors as

$$p(x) = u(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n)$$

where $u \neq 0 \in \mathbb{C}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. (You may use this without proof.) A polynomial $p(x)$ has *repeated roots* if two of the numbers α_i are equal; otherwise, $p(x)$ has *no repeated roots*.

Question 5. Let $R = \mathbb{Z}/5^\infty\mathbb{Z}$ (just like the ring $\mathbb{Z}/10^\infty\mathbb{Z}$ from HW6, but in base 5 instead).
(I'll assume that you solved all of HW6, even if that's not the case.)

(a) Prove that $R = \mathbb{Z}/5^\infty\mathbb{Z}$ is a domain.

(b) Describe which elements of $R = \mathbb{Z}/5^\infty\mathbb{Z}$ are units. (Proof not required.)

(c) Let K be the fraction field[3] of the domain $R = \mathbb{Z}/5^\infty\mathbb{Z}$. Give a **concrete** description of K (along the lines of our original description of $\mathbb{Z}/10^\infty\mathbb{Z}$ on HW6).

What are its elements? What are the operations of addition/multiplication on these elements?

Can one easily see from your description that every nonzero element is invertible, or is that difficult to see? (could be either, depending on your description).

Sketch a proof that your description is correct.

Question 6. Suppose that R is a commutative ring which contains $\mathbb{Z}$.

(a) Prove that if $P \subset R$ is a prime ideal of R, then $P \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$.

(b) Part (a) defines a function $\beta\colon \{\text{prime ideals of } R\} \to \{\text{prime ideals of } \mathbb{Z}\}$. Construct an explicit commutative ring R containing $\mathbb{Z}$ such that the image of β is

$$\operatorname{im}\beta = \{\quad (0), (5), (7), (11), (13), \ldots \quad\}$$

i.e. all prime ideals *except* (2) and (3). Prove (or at least sketch a proof) R has this property.

Question 7. (a) Let F be a field, and let $R \subset F$ be a subring with the property that for every $x \in F$, either $x \in R$ or $\frac{1}{x} \in R$ (or both).

Prove that if I and J are two ideals of R, then either $I \subseteq J$ or $J \subseteq I$.

(b) Construct a proper subring $R \subsetneq \mathbb{Q}$ such that for every $x \in \mathbb{Q}$, either $x \in R$ or $\frac{1}{x} \in R$ (or both).

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[3] from lecture Wed May 30, or §7.5 if you can't wait

Question 8. Given an abelian group A, we say the *10-dual* $A^\vee$ is the abelian group of homomorphisms $f\colon A \to \mathbb{Z}/10\mathbb{Z}$ under pointwise addition. (You may use without proof that $A^\vee$ is an abelian group.) We call an abelian group *10-invisible* if $A^\vee = 0$, i.e. if there are no nonzero group homomorphisms $f\colon A \to \mathbb{Z}/10\mathbb{Z}$.

(a*) Compute $A^\vee$ for $A = \mathbb{Z}$, $A = \mathbb{Z}/6\mathbb{Z}$, and $A = \mathbb{Z}/10\mathbb{Z}$.

(b) We know from class (or will soon) that every finitely-generated abelian group A is isomorphic to

$$A \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}$$

for a unique $r \geq 0$ and a unique sequence of positive integers $n_1 \mid n_2 \mid \cdots \mid n_k$.

In terms of this description, which finitely-generated abelian groups A are 10-invisible?

(c) Choose two of the following abelian groups A, and for each, describe as best you can the abelian group $A^\vee$.

(i) $A = \mathbb{Q}$

(ii) $A = \mathbb{Z}[\frac{1}{6}]$

(iii) $A = \mathbb{Q}/\mathbb{Z}$

(iv) $A = \mathbb{Z}/10^\infty\mathbb{Z}$

(d) [Optional, replaces (b)] In some sense the most interesting dual is the $\mathbb{Q}/\mathbb{Z}$-*dual* A^*, namely the abelian group of homomorphisms $f\colon A \to \mathbb{Q}/\mathbb{Z}$ under pointwise addition. Compute A^* for $A = \mathbb{Z}$, $A = \mathbb{Z}/6\mathbb{Z}$, and $A = \mathbb{Z}/10\mathbb{Z}$. Can you compute A^* for any of the groups in (c)? Compute $(\mathbb{Z}^*)^*$.