Homework 8

Math 120 (Thomas Church, Spring 2018)

Due Thursday, May 24 at 11:59pm.

Write up only the **unstarred** exercises and questions below. (Starred questions are valuable and you really should do them, but they will not be collected or graded.)

7.1.26 7.3.29* 7.4.14(a,b,c,d)*

Question 0*. Prove that the ideal $I = (x^2 + 1)$ in $\mathbb{R}[x]$ is maximal. (For maximum understanding, try to prove this with the same approach we used in class for the ideal $(x - 2, y - 3)$ in $\mathbb{R}[x, y]$.)

Question 1. Let $R \subset \mathbb{R}[x]$ be the subring of $\mathbb{R}[x]$ consisting of polynomials whose coefficient of x is 0:

$$\begin{aligned} R &= \left\{ \ f(x) = a_0 \qquad\quad + a_2x^2 + \cdots + a_nx^n \ \middle|\ a_i \in \mathbb{R} \ \right\} \\ \mathbb{R}[x] &= \left\{ \ f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \ \middle|\ a_i \in \mathbb{R} \ \right\} \end{aligned}$$

You may use without proof that if $g(x)$ and $h(x)$ are polynomials in $\mathbb{R}[x]$, then $\deg(gh) = \deg(g) + \deg(h)$.

Exhibit an ideal $I \subset R$ in R that is not principal, and justify your answer by *proving* that I is not a principal ideal of R.

Question 2. Let $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$.

(a) Find a prime ideal $P_2 \subset R$ such that $P_2 \cap \mathbb{Z} = 2\mathbb{Z}$.

(b) Find a prime ideal $P_3 \subset R$ such that $P_3 \cap \mathbb{Z} = 3\mathbb{Z}$.

(c) Find a prime ideal $P_5 \subset R$ such that $P_5 \cap \mathbb{Z} = 5\mathbb{Z}$.

Justify your answers. For each one, describe (as best you can) the domain R/P.

Question 3. Construct a commutative ring L with the property that for every commutative ring R,

$$\begin{aligned} &\text{the \# of ring homomorphisms } \varphi\colon L \to R \\ \text{is equal to} \quad &\text{the number of elements } r \in R \text{ satisfying } r^2 = 2. \end{aligned}$$

Note that "2" here means the element $1 + 1 \in R$. (You do not have to prove your answer is correct.)

Question 4. Construct a commutative ring M with the property that for every commutative ring R,

$$\begin{aligned} &\text{the \# of ring homomorphisms } \varphi\colon M \to R \\ \text{is equal to} \quad &\text{the number } \left|R^\times\right| \text{ of invertible elements in } R. \end{aligned}$$

Prove your answer is correct.

Question 5. (Hard, Optional) Can there exist a commutative ring N with the property that for every commutative ring R,

$$\begin{array}{rl} & \text{the \# of ring homomorphisms } \varphi\colon N \to R \\ \text{is equal to} & \text{the \# of elements } r \in R \text{ such that both } r \text{ and } 1-r \text{ are units.} \end{array}$$

Either construct such a ring and prove that your answer is correct (at least outline a proof), or prove that no such ring can exist.

In Question 6, you can use the following fact, which we will prove later in the course:

If G is a finitely generated abelian *group, then every subgroup of G is finitely generated.*
(This is false if G is a finitely generated non*abelian group, as you proved for $G = F_2$ in Q5B on HW3.)*

Question 6. Given a complex number $z \in \mathbb{C}$, let $A(z)$ denote the *additive* subgroup of $\mathbb{C}$ generated by the positive powers $1, z, z^2, z^3, \ldots$ under addition.
For example, $A(2) = \langle 1, 2, 4, 8, \ldots \rangle = \mathbb{Z}$, whereas $A(\frac{2}{3}) = \langle 1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots \rangle = \{\frac{p}{3^k} \in \mathbb{Q}\}$.

A complex number $z \in \mathbb{C}$ is called *integral* if $A(z)$ is finitely generated as a group under addition.

(a*) Prove that a rational number $x \in \mathbb{Q}$ is integral if and only if $x \in \mathbb{Z}$.

(b) Describe exactly which elements of $\mathbb{Q}(i)$ are integral. (Recall that $\mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\}$.)

(c) Describe exactly which elements of $\mathbb{Q}(\sqrt{3})$ are integral. (Recall that $\mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$.)

(d) Describe exactly which elements of $\mathbb{Q}(\sqrt{5})$ are integral. (Recall that $\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$.)

(e) Let $x \in \mathbb{C}$ be an integral element, and let $y \in \mathbb{C}$ be an nth root of x (meaning $y^n = x$).
Prove that y is integral.

You can certainly stop here if you want. But if you want to keep thinking about this, you could do some of the following Hard questions. If you do part (f), it can replace part (e). If you do parts (g), (h), or (i), each of these can replace one of (b), (c), or (d). If you do part (j), it can replace part (i).

(f) Prove that if $x \in \mathbb{C}$ and $y \in \mathbb{C}$ are both integral, then $x + y$ and xy are integral. Conclude that the set $\mathbf{A} \subset \mathbb{C}$ of all integral elements of $\mathbb{C}$ forms a subring of $\mathbb{C}$.

(g) Describe exactly which elements of $\mathbb{Q}(\sqrt[3]{2})$ are integral. $\mathbb{Q}(\sqrt[3]{2}) = \{a + b\sqrt[3]{2} + c\sqrt[3]{2}^2 \mid a, b, c \in \mathbb{Q}\}$.

(h) Describe exactly which elements of $\mathbb{Q}(\sqrt[3]{10})$ are integral. $\mathbb{Q}(\sqrt[3]{10}) = \{a + b\sqrt[3]{10} + c\sqrt[3]{10}^2 \mid a, b, c \in \mathbb{Q}\}$.

(i) Prove that $2\cos(\frac{2\pi}{n})$ is integral for any $n \in \mathbb{N}$.

(j) For $z = 2\cos(\frac{2\pi}{n})$, the group $A(z)$ is isomorphic to $\mathbb{Z}^k$ for some rank $k = k(n)$ depending on n.
Compute the rank $k(n)$ for $n = 3, 4, 5, 6, 7$.
Can you express the rank $k(n)$ in general as a function of n?