Homework 7

Math 120 (Thomas Church, Spring 2018)

Due Friday, May 18 at 11:59pm

Note: Remember that according to our definitions (but unlike the book's):

- "R is a subring of F" implies that $1 \in R$.
- "$f\colon S \to R$ is a ring homomorphism" implies that $f(1) = 1$.

Question 0*. You do not have to to write this question up or turn it in, but I encourage you to work it out and make sure you understand why this is a ring.

Let X be any nonempty set, and let $\mathcal{P}(X)$ be the set of all subsets of X (the *power set* of X). Define operations of addition and multiplication on $\mathcal{P}(X)$ by

$$A + B \stackrel{\text{def}}{=} (A - B) \cup (B - A) \qquad\qquad A \times B \stackrel{\text{def}}{=} A \cap B$$

i.e. addition is the symmetric difference of subsets and multiplication is intersection of subsets.

Prove that $\mathcal{P}(X)$ is a commutative ring under these operations.

Question 1. Let F be a field, and let $R \subset F$ be a subring of F. Prove that R is a domain.

Question 2. Suppose that R is a domain, and $x \in R$ satisfies $x^2 = 1$. Prove that $x = 1$ or $x = -1$.

Question 3. Construct a ring K with the property that for every ring R,

$$\text{the \# of ring homomorphisms } \varphi\colon K \to R \qquad \text{is equal to the cardinality } |R|.$$

Question 4. An element $r \in R$ is called *idempotent* if $r^2 = r$.

(a*) Let A and B be commutative rings. Check that in the product ring $A \times B$, the element $(1, 0) \in A \times B$ is idempotent. (You do not need to write this up.)

(b) (Hard) Prove that if R is commutative and $x \in R$ is an idempotent with $x \neq 0$ and $x \neq 1$, then there exist commutative rings A and B such that $R \simeq A \times B$. (How can you find them?)

(c) Prove that if A and B are domains, the ring $R = A \times B$ contains exactly 4 idempotents. *(Hey, isn't there another ring you recently proved has four idempotents?)*

(d) (Hard, Optional) If R is the ring of infinite-integers from HW4, find domains A and B such that $R \simeq A \times B$. Can you describe A and B explicitly? How much can you say about them? In what ways are they like R, or different from R?

Question 5. Let $C([0,1])$ denote the ring of continuous real-valued functions $f\colon [0,1] \to \mathbb{R}$ on the interval $[0,1]$. We can similarly define

$$C\big((0,\tfrac{1}{2})\big) = \{\, f\colon (0,\tfrac{1}{2}) \to \mathbb{R} \;\big|\; f \text{ is continuous} \,\}$$
$$C(\mathbb{R}) = \{\, f\colon \mathbb{R} \to \mathbb{R} \;\big|\; f \text{ is continuous} \,\}$$
$$C(\mathbb{Q}) = \{\, f\colon \mathbb{Q} \to \mathbb{R} \;\big|\; f \text{ is continuous} \,\}$$

and so on. Really, for any $X \subseteq \mathbb{R}$ we can define the ring $C(X)$ of continuous real-valued functions on X:

$$C(X) = \{\, f\colon X \to \mathbb{R} \;\big|\; f \text{ is continuous} \,\}$$

The ring structure comes from pointwise addition and multiplication: the functions $g = f_1 + f_2$ and $h = f_1 \cdot f_2$ are defined by

$$g(x) = f_1(x) + f_2(x) \qquad\qquad h(x) = f_1(x) \cdot f_2(x)$$

(You may assume without proof that $C(X)$ is a ring.)

Recall from elementary school that if $f\colon \mathbb{R} \to \mathbb{R}$ is a function on the whole real line, we can *restrict* f to a smaller set such as $[0,1]$ to obtain $f|_{[0,1]}\colon [0,1] \to \mathbb{R}$.

In fact, for any $X \subsetneq Y$, we can restrict functions $f\colon Y \to \mathbb{R}$ to obtain a function $f|_X\colon X \to \mathbb{R}$. If we write $r(f) = f|_X$, this defines a restriction map $r\colon C(Y) \to C(X)$.

(You may assume without proof that $r\colon C(Y) \to C(X)$ is a ring homomorphism.)

(a) Give an example of two sets $X \subsetneq Y \subseteq \mathbb{R}$ such that $r\colon C(Y) \to C(X)$ is surjective.

(b) Give an example of two sets $X \subsetneq Y \subseteq \mathbb{R}$ such that $r\colon C(Y) \to C(X)$ is not surjective.

(c) Give an example of two sets $X \subsetneq Y \subseteq \mathbb{R}$ such that $r\colon C(Y) \to C(X)$ is injective.

(d) Give an example of two sets $X \subsetneq Y \subseteq \mathbb{R}$ such that $r\colon C(Y) \to C(X)$ is not injective.

(e) (Optional) Is it possible to find two sets $X \subsetneq Y \subseteq \mathbb{R}$ such that $r\colon C(Y) \to C(X)$ is an isomorphism? Either give an example or sketch a proof that it is impossible.

In the questions above, you do not need to formally prove that your example works (since that making it formal might require knowing more topology than we do), but please indicate how your example works. For instance, for your example in (b) where $r\colon C(Y) \to C(X)$ is supposed to be non-surjective, indicate a function $f \in C(X)$ that you believe is not in the image.