Homework 6
Math 120 (Thomas Church, Spring 2018)
Due Thursday, May 10 at 11:59pm.

You should not be looking online or in textbooks to solve these homework problems. However, you are welcome (indeed encouraged) to use a calculator on this homework where necessary. There is a nice arbitrary-precision calculator at apfloat.appspot.com.

On this homework, you can and should keep your arguments informal, i.e. you don’t need to use a bunch of symbols, as long as the underlying argument is clear and rigorous.

Given that you have two weeks, everyone should try to solve at least one of the hard problems (and hopefully more!). Moreover, for every hard problem you solve, you can skip writing up one of the normal problems (though you should probably do them first anyway).

Let \( R \) denote the set of infinite-integers. For example, here are some elements of \( R \):

\[
\begin{align*}
  a &= \cdots 000000001 \\
  b &= \cdots 000000021 \\
  c &= \cdots 000000049 \\
  d &= \cdots 123123123 \\
  e &= \cdots 593593593 \\
  f &= \cdots 999999999 \\
  g &= \cdots 562951413 \quad \text{(digits of } \pi, \text{ backwards)}
\end{align*}
\]

Note that natural numbers like 1 or 21 or 49 show up as elements of \( R \) (namely \( a, b, \) and \( c \) above), padded on the left by 0s. But there are other elements of \( R \) where the digits continue to the left without end, such as \( d, e, \) and \( f \). The element \( g \in R \) is an example showing that the digits don’t have to be periodic or follow any recognizable pattern.

We can add infinite-integers using the usual formula for addition, carrying if necessary. For example:

\[
\begin{array}{cccccc}
 a & + & c & = & a + c & = \\
 \cdots 000000001 & + & \cdots 000000049 & = & \cdots 000000050 \\
 = & = & = & = & = & =
\end{array}
\]

\[
\begin{array}{cccccc}
 d & + & e & = & d + e & = \\
 \cdots 123123123 & + & \cdots 593593593 & = & \cdots 716716716 \\
 = & = & = & = & = & =
\end{array}
\]

You can use without proof that addition is commutative and associative.

Question 1: Compute \( a + f, c + f, \) and \( d + f \).

Question 2: Find an element \( h \in R \) such that \( d + h = \cdots 000000000 \). Show that for any element \( x \in R \), there exists some \( y \in R \) such that \( x + y = \cdots 000000000 \).
We can also multiply elements of $R$ using the usual formula for multiplication:

\[
\begin{array}{rcl}
d & \times & 123123123 \\
\times b & \times & \cdots 00000021 \\
= & \cdots & 123123123 \\
+ & \cdots & 46246246 \\
= & \cdots & 585585583 \\
\end{array}
\]

\[
\begin{array}{rcl}
d & \times & \cdots 3123123 \\
\times d & \times & \cdots 3123123 \\
= & \cdots & 9369369 \\
+ & \cdots & 246 \\
= & \cdots & 23 \\
\end{array}
\]

\[
\begin{array}{rcl}
d & \times & 123123 \\
\times d & \times & 123123 \\
= & \cdots & 585585583 \\
+ & \cdots & 246 \\
= & \cdots & 23 \\
\end{array}
\]

\[
= \cdots \underline{7273129}
\]

Hint: to compute $d \times d$ more quickly, you can just type $123123 \times 123123$ into a calculator to get

\[
\begin{array}{rcl}
123123 \\
\times & 123123 & \\
= & 15159\underline{273129}
\end{array}
\]

Note that only the last six digits are correct; in fact, the next few digits of $d \times d$ are $\cdots \underline{417273129}$. If you want more correct digits in the output, you need more digits in the input:

\[
\begin{array}{rcl}
3123123 & \times & 23123123 \\
23123123 & \times & 23123123 \\
= & 975389\underline{7273129} & = 5346788\underline{17273129} & = 15159303\underline{417273129}
\end{array}
\]

(You can use without proof that multiplication is associative and commutative, and distributes over addition.)

Question 3: Find an element $s \in R$ with the property that

\[
\begin{array}{rcl}
s & \times & 000003 \\
= & \cdots & 000001
\end{array}
\]

In other words, thinking of natural numbers $n \in \mathbb{N}$ as elements of $R$, we’re looking for a solution to the equation $s \times 3 = 1$ in $R$, i.e. a multiplicative inverse of 3 in $R$.

Question 4: Show that 2 does not have a multiplicative inverse in $R$; that is, there is no element $t \in R$ satisfying $t \times 2 = 1$.

Question 5: (Hard) Which natural numbers $n \in \mathbb{N}$ have a multiplicative inverse in $R$? Can you prove it? (Can you describe which $x \in R$ have a multiplicative inverse in $R$?)
As you know, $0 \times 0 = 0$ and $1 \times 1 = 1$. In other words, if we write $t^2$ for $t \times t$, this says $0$ and $1$ are solutions to the equation $t^2 = t$.

Question 6: (Hard) Find two other elements $x \in R$ and $y \in R$ satisfying $x^2 = x$ and $y^2 = y$.

Question 7: (Hard) Can you prove the equation $t^2 = t$ has only four solutions in $R$? (Further thought: how about $t^5 = t$; does this have more solutions than you expect?)

Question 8: (Hard) Find two nonzero elements $a \in R$ and $b \in R$ whose product is zero: $a \neq 0$ and $b \neq 0$, but $a \times b = 0$.

Question 9: Prove that there is no element $x \in R$ satisfying $x^2 = 7$.

Question 10: (Hard) Prove that there is at least one solution $z \in R$ to the equation $z^3 = 7$. 