Homework 6

Math 120 (Thomas Church, Spring 2018)

Due Thursday, May 10 at 11:59pm.

You should **not be looking online or in textbooks** to solve these homework problems. However, you are welcome (indeed encouraged) to use a *calculator* on this homework where necessary. There is a nice arbitrary-precision calculator at `apfloat.appspot.com`.

On this homework, you can and should keep your arguments informal, i.e. you don't need to use a bunch of symbols, as long as the underlying argument is clear and rigorous.

Given that you have two weeks, everyone should try to solve **at least one** of the hard problems (and hopefully more!). Moreover, for every hard problem you solve, you can **skip writing up one of the normal problems** (though you should probably do them first anyway).

Let R denote the set of *infinite-integers*. For example, here are some elements of R:

$$a = \cdots 000000001$$
$$b = \cdots 000000021$$
$$c = \cdots 000000049$$
$$d = \cdots 123123123$$
$$e = \cdots 593593593$$
$$f = \cdots 999999999$$
$$g = \cdots 562951413 \qquad \text{(digits of } \pi\text{, backwards)}$$

Note that natural numbers like 1 or 21 or 49 show up as elements of R (namely a, b, and c above), padded on the left by 0s. But there are other elements of R where the digits continue to the left without end, such as d, e, and f. The element $g \in R$ is an example showing that the digits don't have to be periodic or follow any recognizable pattern.

We can add infinite-integers using the usual formula for addition, carrying if necessary. For example:

$$\begin{array}{r} a \\ +\,c \\ \hline = \end{array} \qquad \begin{array}{r} {\scriptstyle 1} \\ \cdots 000000001 \\ +\ \cdots 000000049 \\ \hline \cdots 000000050 \end{array} \qquad\qquad \begin{array}{r} a \\ +\,d \\ \hline = \end{array} \qquad \begin{array}{r} \cdots 000000001 \\ +\ \cdots 123123123 \\ \hline \cdots 123123124 \end{array}$$

$$\begin{array}{r} d \\ +\,e \\ \hline = \end{array} \qquad \begin{array}{r} {\scriptstyle 1\ \ 1\ \ 1\ \ \ \ } \\ \cdots 123123123 \\ +\ \cdots 593593593 \\ \hline \cdots 716716716 \end{array} \qquad\qquad \begin{array}{r} e \\ +\,g \\ \hline = \end{array} \qquad \begin{array}{r} {\scriptstyle 1\ 11\ 11\ \ \ } \\ \cdots 593593593 \\ +\ \cdots 562951413 \\ \hline \cdots 156545006 \end{array}$$

You can use without proof that addition is commutative and associative.

Question 1: Compute $a + f$, $c + f$, and $d + f$.

Question 2: Find an element $h \in R$ such that $d + h = \cdots 000000000$. Show that for any element $x \in R$, there exists some $y \in R$ such that $x + y = \cdots 000000000$.

We can also *multiply* elements of R using the usual formula for multiplication:

$$\begin{array}{r} d \\ \times\, b \\ \hline = \end{array} \qquad \begin{array}{rr} & \cdots 123123123 \\ \times & \cdots 000000021 \\ \hline & \cdots 123123123 \\ + & \cdots 46246246 \\ \hline = & \cdots 585585583 \end{array} \qquad\qquad \begin{array}{r} d \\ \times\, d \\ \hline = \end{array} \qquad \begin{array}{rl} & \cdots 3123123 \\ \times & \cdots 3123123 \\ \hline & \cdots 9369369 \\ & \cdots 246246 \\ & \cdots 23123 \\ & \cdots 9369 \\ & \cdots 246 \\ & \cdots 23 \\ + & \cdots 9 \\ \hline = & \cdots 7273129 \end{array}$$

Hint: to compute $d \times d$ more quickly, you can just type 123123×123123 *into a calculator to get*

$$\begin{array}{rr} & 123123 \\ \times & 123123 \\ \hline = & 15159\boxed{273129} \end{array}$$

*Note that **only the last six digits are correct**; in fact, the next few digits of $d \times d$ are* $\cdots \mathbf{41}7273129$. *If you want more correct digits in the output, you need more digits in the input:*

$$\begin{array}{rr} & 3123123 \\ \times & 3123123 \\ \hline = & 975389\boxed{7273129} \end{array} \qquad \begin{array}{rr} & 23123123 \\ \times & 23123123 \\ \hline = & 5346788\boxed{\mathbf{1}7273129} \end{array} \qquad \begin{array}{rr} & 123123123 \\ \times & 123123123 \\ \hline = & 15159303\boxed{\mathbf{41}7273129} \end{array}$$

(You can use without proof that multiplication is associative and commutative, and distributes over addition.)

Question 3: Find an element $s \in R$ with the property that

$$\begin{array}{rr} & s \\ \times & \cdots 000003 \\ \hline = & \cdots 000001 \end{array}$$

In other words, thinking of natural numbers $n \in \mathbb{N}$ as elements of R, we're looking for a solution to the equation $s \times 3 = 1$ in R, i.e. a *multiplicative inverse* of 3 in R.

Question 4: Show that 2 does not have a multiplicative inverse in R; that is, there is no element $t \in R$ satisfying $t \times 2 = 1$.

Question 5: (Hard) Which natural numbers $n \in \mathbb{N}$ have a multiplicative inverse in R? Can you prove it? (Can you describe which $x \in R$ have a multiplicative inverse in R?)

As you know, $0 \times 0 = 0$ and $1 \times 1 = 1$.
In other words, if we write t^2 for $t \times t$, this says 0 and 1 are solutions to the equation $t^2 = t$.

Question 6: (Hard) Find two other elements $x \in R$ and $y \in R$ satisfying $x^2 = x$ and $y^2 = y$.

Question 7: (Hard) Can you prove the equation $t^2 = t$ has only four solutions in R?
(Further thought: how about $t^5 = t$; does this have more solutions than you expect?)

Question 8: (Hard) Find two nonzero elements $a \in R$ and $b \in R$ whose product is zero:
$a \neq 0$ and $b \neq 0$, but $a \times b = 0$.

Question 9: Prove that there is no element $x \in R$ satisfying $x^2 = 7$.

Question 10: (Hard) Prove that there *is* at least one solution $z \in R$ to the equation $z^3 = 7$.