Homework 4

Math 120 (Thomas Church, Spring 2018)

Due Thursday, April 26 at 11:59pm.

Write up only the **unstarred** exercises and questions below. (Starred questions are valuable and you really should do them, but they will not be collected or graded.)

4.2.8

4.3.10* 4.3.11*

5.1.1* 5.1.5*

Recall that F_n denotes a free group on n elements.

Question 1. *In at most two sentences*, prove that F_2 is not isomorphic to F_3.

Question 2. Given a group G, the *center* of G is the subgroup

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}$$

of elements that commute with every element of G. The center $Z(G)$ is an abelian subgroup of G. (You may assume this without proof, but you should understand why it is true.)

(a) Prove that $Z(G)$ is a normal subgroup of G.

(b) Prove that if the quotient group $G/Z(G)$ is cyclic, then G is abelian.

(c) (Optional) Is it true that if the quotient group $G/Z(G)$ is abelian, then G is abelian? Prove or give a counterexample.

Question 3. Give a presentation for $G = \mathbb{Z} \times \mathbb{Z}$; i.e. fill in the blanks:

$$\mathbb{Z} \times \mathbb{Z} \cong \langle \ \_\_\_\_\_ \mid \_\_\_\_\_\_\_\_\_ \ \rangle$$

(You do not have to prove your presentation is correct.)

Question 4. (Optional) Prove that S_3 has the presentation

$$S_3 \cong \langle a, b \mid a^2 = 1, b^2 = 1, ababab = 1 \rangle$$

(Double Optional) What additional relation(s) do you need to add to the following to get a presentation of S_4?

$$S_4 \cong \langle a, b, c \mid a^2 = 1, b^2 = 1, c^2 = 1, ababab = 1, bcbcbc = 1, \_\_\_\_\_ \rangle$$

Question 5. Do one of the following three questions on free groups, all marked "Hard".[1] If you solve more than one, please turn them all in; the additional problems will count as optional.

Question 5A. (Hard)
Let $F_2 = \langle a, b \rangle$ be the free group on two generators. Let $\varphi\colon F_2 \to Z_2 \times Z_2$ be the homomorphism satisfying

$$\varphi(a) = (x, 1) \text{ and } \varphi(b) = (1, x)$$

and let $K = \ker \varphi$ be its kernel.

a) Describe which elements of F_2 lie in the subgroup K.

b) The group K itself is a free group of rank k for some $k \in \mathbb{N}$ (meaning it is freely generated by k elements; another way to define the rank is that $n(K, G) = |G|^k$ for all groups G).

   What is its rank k? Can you give a set of k generators for K?
   (If you can't find its rank exactly, perhaps you can bound it above or below?)

Question 5B. (Hard)
Let N denote the commutator subgroup of $F_2 = \langle a, b \rangle$. Prove that N is not finitely generated.
(You defined the commutator subgroup in Exercise 3.1.41 on HW3.)

Question 5C. (Hard)
Consider the 2×2 matrices $x = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ in $\mathrm{GL}_2 \mathbb{R}$.

Prove that the group G generated by x and y is a free group of rank 2 on the generators x and y.

[1] Recall that "*Hard*" means that these questions are especially valuable to figure out on your own; that neither Prof. Church nor Xiaoyu will give help with hard questions, except Prof. Church can clarify confusions about what the question is asking; and that if you're only able to partially solve a hard problem, you shouldn't feel *too* bad, and you should definitely turn in your partial solution.

Question 6. No one's solution to Q6 on HW3 was perfect. By Sunday night, Prof. Church will post one of the following codes on Canvas as a comment on your submission of HW 3C. Do the corresponding assignment below.

Code A: your treatment of the homomorphism was fine, so you don't need to re-do anything there. You gave the correct definition of the elements of the group (at least mostly). However, you did not prove that these elements actually form a group.

A1: State precisely what the elements of your group are. (If the definition you gave previously was not correct, point out what the issues with it were.)

A2: State a precise definition of the group operation on your group.

A3: Give precise statements of the lemmas that you would need to prove to show that this operation is well-defined and that it makes your set into a group. (You do not need to prove these lemmas, as long as you give a precise statement of what properties you would need to check.)

Code B: You stated a few properties of your desired group K, but you did not give enough information to actually define a specific group.

B1: Give precise statements of the properties that your group K would need in order to prove that there is a homomorphism $K \to G$ for every pair of elements $x \in G$ and $y \in G$. (You do not need to prove these properties, as long as you give a precise statement of what properties you need.)

B2: Using these properties, give a complete and correct proof that the map $K \to G$ you define is actually a homomorphism.

Code C: You did not do Question 4.

Make sure you understand the discussion of free groups from class and/or Chapter 6.3. Then do B1 and B2 above.

Code D: You said (and claimed to prove) that $K = \mathbb{Z} \times \mathbb{Z}$ was a correct solution to the problem.

D0: Feel ashamed that you turned in a proof that you should have known was wrong, if you had been careful about what you said. [not to turn in]

D1: Turn in your proof again, along with clear explanations of precisely where the mistakes in your proof were. (It is not enough to just say "well, here's why $\mathbb{Z} \times \mathbb{Z}$ doesn't work"; I want to know where *your* proof went wrong.)

D2: Then do B1 and B2 above.

HW 4A:	4.2.8, Q2, Q4
HW 4B:	Q1, Q3, Q5
HW 4C:	Q6