Homework 3

Math 120 (Thomas Church, Spring 2018)

Due Thursday, April 19 at 11:59pm.

Write up only the **unstarred** exercises and questions below. (Starred questions are valuable and you really should do them, but they will not be collected or graded.)

1.3.1* 1.3.7*

3.1.6* 3.1.7* 3.1.8* 3.1.9* **3.1.41**

Question 1. Let $T \subset S_n$ be the *set* of transpositions. (A transposition is a permutation of the form $(i\ j)$, which swaps two elements and fixes all others. Note that $|T| = \binom{n}{2}$.)
Prove that the symmetric group S_n is generated by T.

Question 2. Let G be a finite group of order $|G| = n$. Prove that there exists a subgroup H of S_n which is isomorphic to G.

Question 3. Recall that a group G is finitely generated if there exists a finite subset $T \subset G$ such that $G = \langle T \rangle$.

(a*) Prove that every finite group is finitely generated.

(b*) Prove that $\mathbb{Z}$ is finitely generated.

(c) Prove that every finitely generated subgroup of $\mathbb{Q}$ is cyclic.

(d) Prove that $\mathbb{Q}$ is not finitely generated.

Question 4. Let G be a finite group of order $|G| = n$, and suppose that p is a prime number dividing n. In this question you will prove that G has an element z of order $|z| = p$. Let

$$S = \left\{ (g_1, g_2, \ldots, g_p) \;\middle|\; g_1 \cdot g_2 \cdots g_p = 1 \right\}$$

be the set of p-tuples of group elements whose product is equal to 1.

(a) Show that $|S| = |G|^{p-1}$. (Since $|G|$ is divisible by p by assumption, (a) implies that $|S|$ is divisible by p.)

Consider the equivalence relation on S defined by $\alpha \sim \beta$ if β is obtained by "rotating" α; in other words, for some k

$$\begin{aligned} \alpha &= (x_1, x_2, \ldots, \qquad\qquad\qquad x_p) \\ \text{and} \quad \beta &= (x_k, x_{k+1}, \ldots, x_p, x_1, \ldots, x_{k-1}) \end{aligned}$$

(b*) Convince yourself that this is an equivalence relation.

(c) Prove that every equivalence class has size 1 or p (using that p is a prime). Conclude that $|S| = a + pb$, where a is the number of classes of size 1 and b is the number of classes of size p.

(d) Show that an equivalence class contains a single element if and only if that element is of the form $(x, x, \ldots, x)$ with $x^p = 1$.

(e) Finish the proof (i.e. prove that G contains an element of order p) by showing that there must be at least one class of size 1 besides $(1, 1, \ldots, 1)$, *à la* HW1 Q3A.

Notation: For any groups H and G, write $n(H, G)$ for the number of homomorphisms from H to G.

If you are scared of infinite cardinalities, you can pretend that $n(H, G) \in \{1, 2, 3, \ldots, \infty\}$; that is, you can pretend that all infinite cardinalities are the same. This will not make things easier or harder, it just might make you less scared.

Question 5. Say you are given two groups A and B. Your goal is to find a new group C with the property that

$$\text{for every group } H, \qquad n(H, C) = n(H, A) \cdot n(H, B). \qquad (*)$$

Construct such a group C (it will depend on the groups A and B you are given!) and prove it has the property $(*)$.

Question 6. (Hard)[1] This week I'll tell you that there *does* exist a group K with the property that

$$\text{for every group } G, \qquad n(K, G) = |G|^2. \qquad (**)$$

Construct such a group K and prove it has the property $(**)$.

HW 3A:	3.1.41, Q3, Q5
HW 3B:	Q1, Q2, Q4
HW 3C:	Q6

[1] Starting with this HW some questions will be labeled "*Hard*". This means that this question is especially valuable to figure out on your own. Accordingly neither Prof. Church nor Xiaoyu will give help with hard questions, except Prof. Church can clarify confusions about what the question is asking. Finally, it means that if you're only able to partially solve a hard problem, you shouldn't feel *too* bad, and you should definitely turn in your partial solution.