# Homework 2

Math 120 (Thomas Church, Spring 2018)

Due Thursday, April 12 at 11:59pm.

Do all the following exercises:

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|---|---|---|
| 1.1.12 | 1.1.13 | |
| 1.6.13 | 1.6.14 | 1.6.18 |
| 2.1.8 | | |
| 2.3.1 | | |

For 1.1.12, 1.1.13, and 2.3.1 you do not have to justify your answers.

**Question 1.** Does there exist some group $K$ with the following property?

$$\text{For every group } G, \quad \text{the number of homomorphisms } f\colon K \to G \text{ is equal to the cardinality } |G| \text{ of } G. \qquad (*)$$

Describe such a group $K$ and prove it has the property $(*)$, or prove that no such group $K$ exists.

**Question 2.** Let $S$ be the set $S = \mathbb{Z}/4\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$. Note that in this question we will not really be considering $S$ as a group, mostly just as a set. We'll say that a bijection $g\colon S \to S$ is *adjacency-preserving* if

$$\text{for all } s \in S, \text{ either } g(s+\overline{1}) = g(s) + \overline{1} \text{ or } g(s+\overline{1}) = g(s) - \overline{1}$$

(a) How many adjacency-preserving bijections on $S$ are there?
You do not have to list them all out (although you might want to give them names for the later parts) but you *do* need to justify why your answer is correct.

Let $G$ be the set of adjacency-preserving bijections on $S$. You should convince yourself that $G$ is a group under composition, but you do not have to prove it.

(b) Does $G$ have a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z}$? Prove or disprove.

(c) How many subgroups $H$ of $G$ containing 4 elements are there? (i.e. $|H| = 4$) Justify your answer.

(d) How many elements of $G$ have order 1? order 2? order 3? order 4? order 5? and so on.

(e) (open-ended, optional) Suppose that instead of $S = \mathbb{Z}/4\mathbb{Z}$ we had taken $S' = \mathbb{Z}/100\mathbb{Z}$, and built the group $G'$ of adjacency-preserving bijections on $S'$. If you wanted to describe the structure of $G'$ to a friend, can you come up with a better way than listing out all its elements and saying which ones multiply to what?

**Question 3.** Let $Z_{12} = \langle x \rangle$ and $Z_9 = \langle y \rangle$. (i.e. $x^{12} = 1$ and $y^9 = 1$; see §2.3 for more on $Z_n$.)
For which integers $a \in \mathbb{Z}$ does there exist an homomorphism $f\colon Z_{12} \to Z_9$ with $f(x) = y^a$?
For which integers $a \in \mathbb{Z}$ does there exist *more than one* such homomorphism?

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| HW 2A: | 1.6.13, 1.6.14, 2.1.8, Q1, Q3 |
| HW 2B: | 1.1.12, 1.1.13, 1.6.18, 2.3.1, Q2 |