Math 113: Linear Algebra and Matrix Theory  
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Homework 7  

Due Wednesday, February 27 in class.  

Question 1. Let $V$ be a vector space with $\dim V = n$. Let $U$ be a subspace of $V$ with $\dim U = k$, and assume that $u_1, \ldots, u_k$ is a basis for $U$.  

a) Prove that if $w_1, \ldots, w_k$ is another basis for $U$, then  
$$ w_1 \wedge \cdots \wedge w_k = a \cdot u_1 \wedge \cdots \wedge u_k $$  
for some nonzero $a \in \mathbb{F}$.  

b) Let $W$ be another subspace of $V$, and assume that $w_1, \ldots, w_k$ is a basis for $W$. (We have dropped the assumption from part a) that $w_i \in U$.)  
Prove that if  
$$ w_1 \wedge \cdots \wedge w_k = a \cdot u_1 \wedge \cdots \wedge u_k $$  
for some nonzero $a \in \mathbb{F}$,  
then $U = W$.  
[Hint: start with a basis $v_1, \ldots, v_\ell$ for $U \cap W$, then extend it to a basis $v_1, \ldots, v_n$ for $V$.]  

Question 2. Let $v_1, \ldots, v_n$ be a basis for $V$. We say that an operator $T \in \mathcal{L}(V)$ is “upper-triangular with respect to the basis $v_1, \ldots, v_n$” if  
$$ T(v_i) \in \text{span}(v_1, \ldots, v_i) \text{ for all } i = 1, \ldots, n. $$  
Assume that $T$ is upper-triangular w.r.t. the basis $v_1, \ldots, v_n$, so for each $i$ we can write  
$$ T(v_i) = d_i \cdot v_i + w_i \quad \text{for some } d_i \in \mathbb{F} \text{ and } w_i \in \text{span}(v_1, \ldots, v_{i-1}). $$  

a) Prove that $\det(T) = d_1 \cdot d_2 \cdots \cdot d_n$.  

b) Prove that each number $d_i$ is an eigenvalue of $T$. Note that the vectors $v_i$ are almost certainly not eigenvectors of $T$!  
[Hint: I do not think a direct approach is best here. First think about how you would prove it when $d_i = 0$, then reduce the general case to this.]
Question 3. Let $V$ and $W$ be finite-dimensional vector spaces, and let $S: V \to W$ be a linear transformation. Let $S^\top: W^* \to V^*$ (pronounced “$S$-transpose”) be defined as follows.\footnote{Recall that $V^* = \mathcal{L}(V, \mathbb{F})$.} If $f \in W^*$ is a linear transformation $f: W \to \mathbb{F}$, then $S^\top(f) \in V^*$ is the linear transformation $V \to \mathbb{F}$ defined by

$$S^\top(f)(v) = f(S(v)).$$

(You do not need to prove that $S^\top(f): V \to \mathbb{F}$ is linear, though you should understand why this is true.)

a) Prove that $S^\top$ is a linear transformation from $W^*$ to $V^*$.

b) Let $\text{Transpose}: \mathcal{L}(V, W) \to \mathcal{L}(W^*, V^*)$ be the function defined by

$$\text{Transpose}(S) = S^\top.$$

Prove that $\text{Transpose}$ is a linear transformation from $\mathcal{L}(V, W)$ to $\mathcal{L}(W^*, V^*)$.

c) Prove that $0^\top = 0$ and $I^\top = I$ (this should not be difficult).

d) If $S \in \mathcal{L}(V, W)$ and $R \in \mathcal{L}(W, U)$, prove that

$$(R \circ S)^\top = S^\top \circ R^\top.$$
**Question 5.** Recall from HW6 that a vector $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ is called a *probability vector* if each entry $v_i$ is $\geq 0$, and $v_1 + \cdots + v_n = 1$. A matrix $A \in \text{Mat}_{n \times n}(\mathbb{R})$ is called a *probability matrix* if each column of $A$ is a probability vector.

a) Prove that if $A$ and $B$ in $\text{Mat}_{n \times n}(\mathbb{R})$ are both probability matrices, then their product $AB$ is also a probability matrix. [Hint: there is a smarter solution than just multiplying out the matrices.]

b) Let $T \in \mathcal{L}(\mathbb{R}^n)$, and let $A$ be its matrix (w.r.t. the standard basis $e_1, \ldots, e_n$). Prove that if $A$ is a probability matrix, then 1 is an eigenvalue of $T$.

c) Bonus question, for no points: prove that 1 is the *largest* eigenvalue of $T$. 
