Math 113 Homework 4 Solutions

Solutions by Jenya Sapir, with edits by Tom Church.

Exercises from the book.

Exercise 1 Suppose $T \in \mathcal{L}(V)$. Prove that if $U_1, \ldots, U_m$ are subspaces of V invariant under T, then $U_1 + \cdots + U_m$ is invariant under T.

Proof. We need to show that if $u \in U_1 + \cdots + U_m$ then $Tu \in U_1 + \cdots + U_m$. So let $u \in U_1 + \cdots + U_m$. Then for each $i = 1, \ldots, m$, we can find $u_i \in U_i$ s.t.

$$u = u_1 + \cdots + u_m$$

Since T is a linear map, when we apply it to u, we get

$$Tu = Tu_1 + \cdots + Tu_m$$

Since $U_1, \ldots, U_m$ are T-invariant, and since $u_i \in U_i$ for each i, we have that $Tu_i \in U_i$ for each i. Thus, $Tu_1 + \cdots + Tu_m \in U_1 + \cdots + U_m$. Therefore, $U_1 + \cdots + U_m$ is T-invariant.

□

Exercise 2 Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of any collection of subspaces of V invariant under T is invariant under T.

Proof. Let $\mathcal{U}$ be a collection of subspaces of V invariant under T. Recall that $\bigcap \mathcal{U}$ is the set of vectors that belong to each subspace U in the collection $\mathcal{U}$. We want to show that $\bigcap \mathcal{U}$ is invariant under T.

Let $v \in \bigcap \mathcal{U}$. Then $v \in U$ for each $U \in \mathcal{U}$. Since each U is T-invariant, $Tv \in U$ for each $U \in \mathcal{U}$, as well. Thus, $Tv \in \bigcap \mathcal{U}$. Therefore $\bigcap \mathcal{U}$ is T-invariant.

□

Exercise 4 Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\text{Null}(T - \lambda I)$ is invariant under S for each $\lambda \in \mathbb{F}$.

Proof. Let $\lambda \in \mathbb{F}$. Let $v \in \text{Null}(T - \lambda I)$. Then $(T - \lambda I)(v) = 0$, that is, $Tv = \lambda v$ (so v is an eigenvector of T with eigenvalue λ.) Apply S to both sides of this equality:

$$STv = S(\lambda v)$$

On the left hand side, we get STv which is the same as TSv. On the right we get $S(\lambda v)$, which is the same as $\lambda S(v)$ since S is linear. Thus,

$$T(Sv) = \lambda Sv$$

Therefore, $T(Sv) - Sv = 0$, so $Sv \in \text{Null}(T - \lambda I)$. Thus $\text{Null}(T - \lambda I)$ is S-invariant for each $\lambda \in \mathbb{F}$. □

Exercise 5 Define $T \in \mathcal{L}(\mathbb{F}^2)$ by

$$T(w, z) = (z, w)$$

Find all eigenvalues and eigenvectors of T.

Answer. If (w, z) is an eigenvector of T then there is some λ for which $T(w, z) = \lambda(w, z)$. This means that

$$T(w, z) = (z, w) = (\lambda w, \lambda z).$$

This gives two equations: $z = \lambda w$, and $w = \lambda z$. This implies that $w = 0 \iff z = 0$, so any nonzero eigenvector has $w \neq 0$ and $z \neq 0$. Substituting the second equation into the first, we find that $z = \lambda^2 z$; since $z \neq 0$, this implies that $\lambda^2 = 1$. Therefore the only possible eigenvalues of T are 1 and -1. (In the book, the field $\mathbb{F}$ is always $\mathbb{R}$ or $\mathbb{C}$, so 1 and -1 are distinct elements of $\mathbb{F}$, meaning $1 \neq -1$.)

Consider the vectors $x = (1,1)$ and $y = (1,-1)$. We have

$$T(1,1) = (1,1)$$

and

$$T(1,-1) = (-1,1)$$

So $Tx = x$ and $Ty = -y$. Thus x is a nonzero eigenvector with eigenvalue 1, and y is a nonzero eigenvector with eigenvalue -1. This shows that 1 and -1 are the eigenvalues of T.

The eigenspace of T for the eigenvalue 1 is $\{v \in \mathbb{F}^2 \mid T(v) = v\}$. If $v = (w,z)$, this equation becomes $(z,w) = (w,z)$. This is true if and only if $w = z$. Therefore the eigenspace of T for the eigenvalue 1 is the line $\{\,(w,w) \mid w \in \mathbb{F}\,\} = \{\, wx \mid w \in \mathbb{F}\,\}$.

The eigenspace of T for the eigenvalue -1 is $\{v \in \mathbb{F}^2 | T(v) = -v\}$. If $v = (w,z)$, this equation becomes $(z,w) = -(w,z) = (-w,-z)$. This is true if and only if $z = -w$, since then $w = -z$ as well. Therefore the eigenspace of T for the eigenvalue 1 is the line $\{\,(w,-w) \mid w \in \mathbb{F}\} = \{\, wy \mid w \in \mathbb{F}\,\}$.

We have found that the eigenvalues of T are 1 and -1, and the eigenvectors of T are scalar multiples of $x = (1,1)$ and $y = (1,-1)$. □

Exercise 6 Define $T \in \mathcal{L}(\mathbb{F}^3)$ by

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$$

Find all eigenvalues and eigenvectors of T.

Answer. For this operator T, the eigenvalue equation $T(v) = \lambda v$ takes the form

$$(2z_2, 0, 5z_3) = (\lambda z_1, \lambda z_2, \lambda z_3)$$

This gives three equations: $2z_2 = \lambda z_1$, $0 = \lambda z_2$, and $5z_3 = \lambda z_3$.

Let's first find the set of eigenvectors with eigenvalue 0. Suppose $x = (z_1, z_2, z_3)$ with $Tx = 0x$. Then

$$\begin{aligned} T(z_1, z_2, z_3) &= (2z_2, 0, 5z_3) \\ &= (0,0,0) \end{aligned}$$

implies $z_2 = z_3 = 0$. Since $T(a,0,0) = (0,0,0)$ for all $a \in \mathbb{F}$ the set of eigenvectors with eigenvalue 0 is the set of all scalar multiples of $(1,0,0)$.

Next we find the eigenvectors with eigenvalue 5. Suppose $x = (z_1, z_2, z_3)$ with $Tx = 5x$. Then

$$\begin{aligned} T(z_1, z_2, z_3) &= (2z_2, 0, 5z_3) \\ &= (5z_1, 5z_2, 5z_3) \end{aligned}$$

implies $z_2 = 0$ and $5z_1 = 2z_2$. Thus, $z_1 = 0$ as well. Since $T(0,0,a) = (0,0,5a)$ for all $a \in \mathbb{F}$, the set of eigenvectors with eigenvalue 0 is the set of all scalar multiples of $(0,0,1)$.

Therefore T has eigenvalues 0 and 5, and its set eigenvectors is the set spanned by $(1,0,0)$ together with the set spanned by $(0,0,1)$. □

Exercise 8 Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(\mathbb{F}^\infty)$ defined by

$$T(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$$

Answer. We will show that all $\lambda \in \mathbb{F}$ are eigenvalues of T, and the set of eigenvectors of T with eigenvalue λ is the set $V_\lambda = \{(z, \lambda z, \lambda^2 z, \dots) \mid z \in \mathbb{F}\}$.

First we show that if v is an eigenvector of T, then $v \in V_\lambda$ for some λ. That is, we show that $v = (z, \lambda z, \lambda^2 z, \dots)$ for some z and some λ. Suppose $v = (z_1, z_2, z_3, \dots)$ is an eigenvector for T with eigenvalue λ. Then the eigenvalue equation $T(v) = \lambda v$ takes the form

$$(\lambda z_1, \lambda z_2, \lambda z_3, \dots) = (z_2, z_3, z_4, \dots)$$

Since two vectors in $\mathbb{F}^\infty$ are equal if and only if their terms are all equal, this yields an infinite sequence of equations:

$$z_2 = \lambda z_1, \quad z_3 = \lambda z_2, \dots, \quad z_n = \lambda z_{n-1}, \dots$$

From this, we can repeatedly substitute $z_n = \lambda z_{n-1} = \lambda^2 z_{n-2} = \dots$, so in fact (by a simple induction)

$$z_n = \lambda^{n-1} z_1$$

So every eigenvector v with eigenvalue λ is of the form $v = (z_1, \lambda z_1, \lambda^2 z_1, \dots)$. Furthermore, for any $z \in \mathbb{F}$, if we set $z_1 = z$, $z_2 = \lambda z$, $\dots$, $z_n = \lambda^n z$, the vector

$$v = (z, \lambda z, \lambda^2 z, \dots)$$

satisfies the equations above and is an eigenvector of T with eigenvalue λ Therefore, the eigenspace V_λ of T with eigenvalue λ is the set of vectors

$$V_\lambda = \big\{(z, \lambda z, \lambda^2 z, \dots) \,\big|\, z \in \mathbb{F}\big\}.$$

Finally, we show that every single $\lambda \in \mathbb{F}$ occurs as an eigenvalue of T. Given $\lambda \in \mathbb{F}$, consider the vector $v = (1, \lambda, \lambda^2, \dots)$. Applying T to v, we get

$$\begin{aligned} T(v) = (1, \lambda, \lambda^2, \dots) &= (\lambda, \lambda^2, \lambda^3, \dots) \\ &= \lambda(1, \lambda, \lambda^2, \dots) \end{aligned}$$

Thus $T(v) = \lambda v$ for this vector. We have thus shown that all $\lambda \in \mathbb{F}$ are eigenvalues for T, and the eigenspace for λ is $V_\lambda = \{(z, \lambda z, \lambda^2 z, \dots) \mid z \in \mathbb{F}\}$. □

Exercise 12 Suppose $T \in \mathcal{L}(V)$ is such that every vector V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

Proof. We want to show that there is some $a \in \mathbb{F}$ s.t. $Tv = av$ for all $v \in V$. That is, we want to show that every vector of V has the same eigenvalue.

For any λ, let $V_\lambda = \text{Null}(T - \lambda I)$ be the subspace of V of eigenvectors with eigenvalue λ. Choose any nonzero vector $v \in V$. Since every vector is an eigenvector of T, we must have $T(v) = av$ for some $a \in \mathbb{F}$. We want to show that V_a is in fact all of V.

Suppose $w \notin V_a$. Since every vector of V is an eigenvector of T, $T(w) = bw$ for some $b \in \mathbb{F}$. Likewise, $v + w$ is also an eigenvector of T so there is some c s.t.

$$\begin{aligned} T(v + w) &= cv + cw, \text{ but } T \text{ is linear, so} \\ T(v + w) &= Tv + Tw \\ &= av + bw \end{aligned}$$

Thus,

$$(c-a)v + (c-b)w = 0$$

Then this means that $(c-a)v = (b-c)w$. Suppose $c \neq b$. Then $b-c$ is invertible in $\mathbb{F}$ so we get that w is a multiple of v. But that would imply $w \in V_a$, which is a contradiction.

Suppose that $c = b$. Then we would get that $(c-a)v = 0$. Since $v \neq 0$ this means $c = a$. Thus $b = a$ so w is a vector with eigenvalue a, meaning $w \in V_a$. Again, we get a contradiction.

Therefore, there are no $w \in V$ which are not in V_a. That means that all $w \in V$ have eigenvalue a. This is exactly the statement that $Tw = aw$ for all $w \in V$. That is, T must be a multiple of the identity. □

Question 1. Let $C^\infty(\mathbb{R})$ denote the vector space (over $\mathbb{R}$) of infinitely-differentiable real-valued functions $f : \mathbb{R} \to \mathbb{R}$.

a) Let U denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which vanish at 2 and at 7:

$$U = \{f \in C^\infty(\mathbb{R}) \mid f(2) = 0, f(7) = 0\}$$

Prove that the quotient vector space $C^\infty(\mathbb{R})/U$ is finite dimensional. What is its dimension?

b) Let W denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which "vanish to second order at 0":

$$W = \{f \in C^\infty(\mathbb{R}) \mid f(0) = 0, f'(0) = 0, f''(0) = 0\}$$

Prove that the quotient vector space $C^\infty(\mathbb{R})/W$ is finite dimensional, and find a basis for $C^\infty(\mathbb{R})/W$.

Proof. a) Define the linear transformation $T\colon C^\infty(\mathbb{R}) \to \mathbb{R}^2$ by $T(f) = \big(f(2), f(7)\big)$.[1] The kernel of T is

$$\begin{aligned} \ker T &= \{f \in C^\infty(\mathbb{R}) \,|\, T(f) = 0\} \\ &= \{f \in C^\infty(\mathbb{R}) \,|\, f(2) = 0, f(7) = 0\} \\ &= U. \end{aligned}$$

The Quotient Isomorphism Theorem thus tells us that $\overline{T}\colon C^\infty(\mathbb{R})/U \to \text{Image } T$ is an isomorphism, so we need to understand Image T.

Choose two functions $f, g \in C^\infty(\mathbb{R})$ that satisfy $T(f) = (1, 0)$ and $T(g) = (0, 1)$, such as:[2]

$$\begin{aligned} f(x) &= \frac{7-x}{5} \\ g(x) &= \frac{x-2}{5} \end{aligned}$$

[1] For example, if $f(x) = x^2$ then $T(f) = (4, 49)$; if $g(x) = e^x$ then $T(g) = (e^2, e^7)$, if $h(x) = \sin x$ then $T(h) = (\sin 2, \sin 7)$, etc.

[2] Many other choices are possible, for example:

$$\begin{aligned} f(x) &= \cos^2\left(\frac{\pi}{2}x\right) \\ g(x) &= \sin^2\left(\frac{\pi}{2}x\right) \end{aligned}$$

Since

$$\begin{array}{ll} f(2)=1 & f(7)=0 \\ g(2)=0 & g(7)=1 \end{array}$$

we have $T(f) = (1,0)$ and $T(g) = (0,1)$. This shows that $(1,0) \in \text{Image } T$ and $(0,1) \in \text{Image } T$. Since these are the standard basis vectors $e_1 = (1,0)$ and $e_2 = (0,1)$, they span $\mathbb{R}^2$, and so Image $T = \mathbb{R}^2$.

Since Image$T = \mathbb{R}^2$, the Quotient Isomorphism Theorem states that $\overline{T}\colon C^\infty(\mathbb{R})/U \to \mathbb{R}^2$ is an isomorphism. Since $C^\infty(\mathbb{R})/U$ and $\mathbb{R}^2$ are isomorphic, they have the same dimension: therefore $C^\infty(\mathbb{R})/U$ has dimension 2.

b) Define the linear transformation $S\colon C^\infty(\mathbb{R}) \to \mathbb{R}^3$ by [3]

$$S(f) = \big(f(0), f'(0), f''(0)\big).$$

The kernel of S is

$$\begin{aligned} \ker S &= \{f \in C^\infty(\mathbb{R}) \mid S(f) = 0\} \\ &= \{f \in C^\infty(\mathbb{R}) \mid f(0) = 0, f'(0) = 0, f''(0) = 0\} \\ &= W. \end{aligned}$$

The Quotient Isomorphism Theorem thus tells us that $\overline{S}\colon C^\infty(\mathbb{R})/W \to \text{Image } S$ is an isomorphism, so we need to understand Image S. Consider the following functions in $C^\infty(\mathbb{R})$:

$$\begin{aligned} f_1 &= 1 \\ f_2 &= x - 1 \\ f_3 &= x^2 - 2x + 1 \end{aligned}$$

These three functions are infinitely differentiable, so they are in $C^\infty(\mathbb{R})$. Their only important properties are that

$$\begin{array}{lll} f_1(0)=1 & f_1'(0)=0 & f_1''(0)=0 \\ f_2(0)=0 & f_2'(0)=1 & f_2''(0)=0 \\ f_3(0)=0 & f_3'(0)=0 & f_3''(0)=1 \end{array}$$

This implies that

$$S(f_1) = e_1, \quad S(f_2) = e_2, \quad S(f_3) = e_3.$$

Therefore e_1, e_2, and e_3 are all in Image$S$. Since e_1, e_2, e_3 is a basis for $\mathbb{R}^3$, this shows that Image$S = \mathbb{R}^3$.

Since Image$S = \mathbb{R}^3$, the Quotient Isomorphism Theorem states that $\overline{S}\colon C^\infty(\mathbb{R})/W \to \mathbb{R}^3$ is an isomorphism. Since $C^\infty(\mathbb{R})/W$ and $\mathbb{R}^3$ are isomorphic, they have the same dimension: therefore $C^\infty(\mathbb{R})/W$ has dimension 3.

Consider the elements $v_1 = f_1 + W$, $v_2 = f_2 + W$, and $v_3 = f_3 + W$ in the quotient space $C^\infty(\mathbb{R})/W$. We will show they are linearly independent. Assume that $av_1 + bv_2 + cv_3 = 0$ in $C^\infty(\mathbb{R})/W$. The above formula shows that

$$\overline{S}(v_1) = \overline{S}(f_1 + W) = e_1, \quad \overline{S}(v_2) = \overline{S}(f_2 + W) = e_2, \quad \overline{S}(v_3) = \overline{S}(f_3 + W) = e_3.$$

Since $\overline{S}$ is linear, $\overline{S}(av_1 + bv_2 + cv_3) = ae_1 + be_2 + ce_3$. But e_1, e_2, e_3 are linearly independent, so we conclude that $a = b = c = 0$. This shows that v_1, v_2, v_3 are linearly independent in the quotient space $C^\infty(\mathbb{R})/W$. Since this vector space has dimension 3, this implies that v_1, v_2, v_3 is a basis for $C^\infty(\mathbb{R})/W$. □

[3] For example, if $f(x) = x^2$ then $S(f) = (0,0,2)$; if $g(x) = e^x$ then $S(g) = (1,1,1)$; if $h(x) = \sin x$ then $S(h) = (0,1,0)$, etc.

Question 2. Let $C^\infty(\mathbb{R},\mathbb{C})$ be the vector space (over $\mathbb{C}$) of complex-valued functions $f:\mathbb{R}\to\mathbb{C}$ that are infinitely differentiable. Let V be the space of functions $f\in C^\infty(\mathbb{R},\mathbb{C})$ satisfying the equation $f''=-f$:

$$V=\{f\in C^\infty(\mathbb{R},\mathbb{C})\mid f''=-f\}$$

- Prove that V is a subspace of $C^\infty(\mathbb{R},\mathbb{C})$.
- Assume without proof that $\dim V\le 2$. Prove that the functions $\sin x$ and $\cos x$ both lie in V, and moreover that $(\sin x,\cos x)$ form a basis for V.
- Let D be the operator on $\mathbb{C}^\infty(\mathbb{R},\mathbb{C})$ defined by $D(f)=f'$. Prove that V is an invariant subspace for D.
- Now consider $D\in\mathcal{L}(V)$ as an operator on V. Find a basis for V consisting of eigenvectors for D. What are their eigenvalues?

Proof.

- First we prove that V is a subspace of $C^\infty(\mathbb{R},\mathbb{C})$. First, note that zero function $z(x)=0$ is infinitely differentiable, and satisfies $z''(x)=0=-z(x)$ for all $x\in\mathbb{R}$. Next, suppose $f,g\in V$. Then $(f+g)''=f''+g''=-f-g$. Thus, $(f+g)''=-(f+g)$, so $f+g\in V$. Lastly, let $a\in\mathbb{C}$. Then $(af)''=af''=-af$, so $af\in V$. Thus V is a subspace of $C^\infty(\mathbb{R},\mathbb{C})$.
- Consider the functions $\sin x$ and $\cos x$. Then $\sin''(x)=(\cos'(x))=-\sin(x)$ and $\cos''(x)=(-\sin(x))'=-\cos(x)$. Thus $\sin(x),\cos(x)\in V$.

  To show that $(\sin x,\cos x)$ form a basis for V, first we show that they are linearly independent. So suppose there are numbers $a,b\in\mathbb{C}$ s.t. $a\sin(x)+b\cos(x)=0$. Then, plugging in $x=0$, we get $b=0$ since $\sin(0)=0$ and $\cos(0)=1$. Plugging in $x=\pi/2$, we get $a=0$ since $\sin(\pi/2)=1$ and $\cos(\pi/2)=0$. Thus $\sin(x)$ and $\cos(x)$ are linearly independent.

  Since $\sin(x)$ and $\cos(x)$ are linearly independent, the dimension of V must be at least 2. Since we were given that $\dim V$ is at most 2, we conclude that $\dim V=2$. Thus $\sin(x)$ and $\cos(x)$ form a basis for V.
- Let D be the operator on $\mathbb{C}^\infty(\mathbb{R},\mathbb{C})$ defined by $D(f)=f'$. To show that V is invariant under D, we must show that if $f\in V$ then $Df\in V$. So suppose that $f\in V$, and set $g=D(f)$. Then $f''=-f$. Differentiating both sides of this equation, we get that $f'''=-f'$, or in other words $g''=-g$. Thus $g=D(f)$ lies in V. Therefore, V is invariant under D.
- Now consider $D\in\mathcal{L}(V)$ as an operator on V. Find a basis for V consisting of eigenvectors for D. What are their eigenvalues?

  The properties $(\sin x)'=\cos x$ and $(\cos x)'=-\sin x$ mean that

  $$D\big(a\sin(x)+b\cos(x)\big)=-b\sin(x)+a\cos(x).$$

  We have seen this linear transformation before, in another guise, as

  $$T\colon\mathbb{R}^2\to\mathbb{R}^2\qquad T(x,y)=(-y,x)$$

  Therefore following the path we took in class, define

  $$f=\cos(x)+i\sin(x)$$
  $$g=\cos(x)-i\sin(x)$$

  Then

  $$D(f)=-\sin(x)+i\cos(x)=if$$

  and

  $$D(g)=-\sin(x)-i\cos(x)=-ig$$

Thus f and g are eigenvectors for D with eigenvalues i and $-i$. Since they have distinct eigenvalues, Theorem 5.6 in the book implies that they are linearly independent. Since $\dim V \leq 2$, any spanning list of length 2 forms a basis for V.

Remark by TC: you have probably learned what the eigenvectors of D as an operator on $\mathbb{C}^\infty(\mathbb{R}, \mathbb{C})$ are in a previous class. For the eigenvalue a, the eigenvalue equation $D(f) = af$ becomes the differential equation $f' = af$, and you may already know that the solutions to this equation are (constant multiples of)

$$f(x) = e^{ax},$$

since the chain rule implies that

$$(e^{ax})' = a \cdot e^{ax}.$$

But the functions f and g you found above are eigenvectors with eigenvalues $a = i$ and $a = -i$, so they must be of the form Ce^{ix} and Ce^{-ix}! We can find the constants by plugging in 0, since $Ce^{i \cdot 0} = C$. By plugging in $f(0) = \cos(0) + i \sin 0 = 1 + i \cdot 0 = 1$ and $g(0) = \cos(0) - i \sin 0 = 1 - i \cdot 0 = 1$ we see that the constants are 1 for both f and g. Therefore you have proved the famous formula of Euler:

$$e^{ix} = \cos x + i \cdot \sin x \qquad e^{-ix} = \cos x - i \cdot \sin x.$$

In particular, if we evaluate the first eigenfunction at π we get $e^{i\pi} = \cos \pi + i \cdot \sin \pi = -1 + i \cdot 0$, or in other words.

$$e^{i\pi} = -1. \qquad \square$$