1. Math 113 Homework 3 Solutions

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Exercises from the book.

Exercise 4 Suppose that T is a linear map from V to $\mathbb{F}$. Prove that if $u \in V$ is not in $\operatorname{Null} T$, then

$$V = \operatorname{Null} T \oplus \{au \mid a \in F\}$$

Proof. To show that $V = \operatorname{Null} T \oplus \{au \mid a \in F\}$, we need to show that $V = \operatorname{Null} T + \{au \mid a \in F\}$ and that $\operatorname{Null} T \cap \{au \mid a \in F\} = \{0\}$.

First we show that $V = \operatorname{Null} T + \{au \mid a \in F\}$. Let $v \in V$. We need to find $n \in \operatorname{Null} T$ and $w \in \{au \mid a \in F\}$ for which $v = n + w$. Suppose $T(v) = a \in F$, and that $T(u) = b \in \mathbb{F}$. We know that $b \neq 0$ because u is not in $\operatorname{Null} T$. Thus, b has an inverse in $\mathbb{F}$. Let $c = ab^{-1} \in \mathbb{F}$. Note that this means c times b gives back a. So,

$$\begin{aligned} T(cu) &= cT(u) \\ &= cb \text{ since T(u) = b} \\ &= a \end{aligned}$$

Thus, $T(cu) = T(v)$. Let $n = v - cu$. Then $T(n) = T(v) - T(cu) = 0$. Therefore, $n \in \operatorname{Null} T$. Set $w = cu$. Then $w \in \{au \mid a \in F\}$, and $v = n + w$. So we can write v as the sum of elements of $\operatorname{Null} T$ and $\{au \mid a \in F\}$. Therefore, $V = \operatorname{Null} T + \{au \mid a \in F\}$.

Next we show that $\operatorname{Null} T \cap \{au \mid a \in F\} = \{0\}$. Suppose $v \in \operatorname{Null} T \cap \{au \mid a \in F\}$. Then $v \in \{au \mid a \in F\}$, so $v = au$ for some $a \in \mathbb{F}$.

Since $v \in \operatorname{Null} T$, $T(v) = 0$. So $T(au) = 0$. But $T(au) = aT(u)$. Since $aT(u) = 0$, either $a = 0$ or $T(u) = 0$. But u is not in $\operatorname{Null} T$, so $T(u) \neq 0$. This means a must equal 0. So $v = au$ implies that $v = 0$. Therefore, $\operatorname{Null} T \cap \{au \mid a \in F\} = \{0\}$.

Since $V = \operatorname{Null} T + \{au \mid a \in F\}$ and that $\operatorname{Null} T \cap \{au \mid a \in F\} = \{0\}$, we have that $V = \operatorname{Null} T \oplus \{au \mid a \in F\}$. ☐

Exercise 8 Suppose V is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{Null} T = \{0\}$ and $\operatorname{Image} T = \{Tu \mid u \in U\}$.

Proof. Proposition 2.13 from the book says that if V is finite dimensional and W is a subspace of V then we can find a subspace U of V for which $V = W \oplus U$. Proposition 3.1 from the book says that $\operatorname{Null} T$ is a subspace of V. Setting $W = \operatorname{Null} T$, we can apply prop. 2.13 to get a subspace U of V for which

$$V = \operatorname{Null} T \oplus U$$

Now we just need to show that $\operatorname{Image} T = \{Tu \mid u \in U\}$. First we show that $\operatorname{Image} T \subset \{Tu \mid u \in U\}$. So let $w \in \operatorname{Image} T$. That means there is some $v \in V$ for which $T(v) = w$. Since $v \in V$ and we have that $V = \operatorname{Null} T \oplus U$, we can find vectors $n \in \operatorname{Null} T$ and $u \in U$ for which $v = n + u$. Thus,

$$\begin{aligned} T(v) &= T(n) + T(u) \\ &= 0 + T(u) \text{ since } n \in \operatorname{Null} T \end{aligned}$$

We had that $w = T(v)$. So, $w = T(u)$ for some $u \in U$. That means that $w \in \{Tu \mid u \in U\}$. Thus $\operatorname{Image} T \subset \{Tu \mid u \in U\}$.

Now we show that $\{Tu \mid u \in U\} \subset \operatorname{Image} T$. But for any element $u \in U$, u is also in V as $U \subset V$. Thus Tu is in the image of T by definition. Therefore $\{Tu \mid u \in U\} \subset \operatorname{Image} T$.

So we have shown that $\operatorname{Image} T = \{Tu \mid u \in U\}$. Thus, there exists a subspace U of V s.t. $V = \operatorname{Null} T \oplus U$ and $\operatorname{Image} T = \{Tu \mid u \in U\}$.

☐

Exercise 9 Prove that if T is a linear map from $\mathbb{F}^4$ to $\mathbb{F}^2$ such that

$$\operatorname{Null} T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_1 = 5x_2, x_3 = 7x_4\}$$

then T is surjective.

Proof. Theorem 3.4 from the book states that since $\mathbb{F}^4$ is finite dimensional then

$$\dim \mathbb{F}^4 = \dim \operatorname{Null} T + \dim \operatorname{Image} T$$

We will show that $\dim \operatorname{Null} T$ is at most 2. To do this, we will show that $\operatorname{Null} T$ has a spanning set with 2 vectors. Let

$$v_1 = (5, 1, 0, 0)$$
$$v_2 = (0, 0, 7, 1)$$

Then $v_1, v_2 \in \operatorname{Null} T$.

Let $x = (x_1, x_2, x_3, x_4) \in \operatorname{Null} T$. We are given that $\operatorname{Null} T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_1 = 5x_2, x_3 = 7x_4\}$. So in fact x is of the form $x = (5x_2, x_2, 7x_4, x_4)$. Thus $x = x_2 v_1 + x_4 v_2$. So v_1 and v_2 span $\operatorname{Null} T$. (In fact, it is true that they form a basis for T, but we don't need that for this problem.) Since $\operatorname{Null} T$ has a spanning set of 2 vectors, $\dim \operatorname{Null} T \leq 2$. (Note that this means $4 - \dim \operatorname{Null} T \geq 2$.)

Since $\dim \mathbb{F}^4 = 4$, Theorem 3.4 gives us

$$4 - \dim \operatorname{Null} T = \dim \operatorname{Image} T \text{ so,}$$
$$2 \leq \dim \operatorname{Image} T$$

Proposition 3.3 in the book says that $\operatorname{Image} T$ is a subspace of $\mathbb{F}^2$. So we have that $\operatorname{Image} T$ is a subspace of $\mathbb{F}^2$ of dimension at least 2. But $\mathbb{F}^2$ has dimension 2. Thus, $\operatorname{Image} T$ has dimension exactly 2. We showed in exercise 11 on the last homework that if a subspace U of a finite dimensional vector space V has the same dimension as V then $U = V$. Thus $\dim \operatorname{Image} T = \dim \mathbb{F}^2$ implies $\operatorname{Image} T = \mathbb{F}^2$. Since $\operatorname{Image} T = \mathbb{F}^2$, we have shown that T is surjective.

□

Exercise 14 Suppose that W is finite dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

Proof. Suppose T is injective. We need to find an $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

Now, for every $u \in \operatorname{Image} T$, there is a $v \in V$ s.t. $T(v) = u$. In fact, v is unique: if $v, v' \in V$ were vectors s.t. $T(v) = u$ and $T(v') = u$, then the fact that T is injective implies that $v = v'$.

We have that $\operatorname{Image} T$ is a subspace of T by prop 3.3 in the book. So by prop 2.13, and the fact that W is finite dimensional, we can find a subspace N of W s.t. $W = \operatorname{Image} T \oplus N$. Thus, any $w \in W$ can be written $w = u + n$ for $u \in \operatorname{Image} T$ and $n \in N$, and u and n are unique.

So, define $S : W \to V$ as follows. For $w \in W$ with $w = u + n$ with $u \in ImT$ and $n \in N$, let $S(w) = v$ where v is the unique vector in V for which $T(v) = u$. Note that this map is well-defined: there is only one way to write w as a sum of an element of $\operatorname{Image} T$ and an element of N, and there is only one element of V for which $T(v) = u$. So given a $w \in W$, there is only one element of V that can be $S(w)$.

We need to show that S is linear, and that ST is the identity. First we show that it is linear. So suppose $w, w' \in W$. Then we can find $u, u' \in \operatorname{Image} T$ and $n, n' \in N$ s.t. $w = u + n$ and $w' = u' + n'$. Note that then $w + w' = u + u' + n + n'$ where $u + u' \in U$ and $n + n' \in N$.

Let $v, v' \in V$ be vectors for which $T(v) = u$ and $T(v') = u'$. So $S(w) = v$ and $S(w') = v'$. If $T(v) = u$ and $T(v') = u'$, the fact that T is linear implies that $T(v + v') = u + u'$. Thus, $S(w + w') = v + v'$. This means that $S(w + w') = S(w) + S(w')$.

Now let $a \in \mathbb{F}$. Then $aw = au + an'$. Since $\operatorname{Image} T, N$ are subspaces of W, $au \in \operatorname{Image} T$ and $an' \in N$. Also, if $T(v) = u$ then $T(av) = au$ because T is linear. So $S(aw) = av$, and $aS(w) = av$. Thus, S is a linear map.

Next we show that ST is the identity map. Let $v \in V$. Then $T(v) = u \in \operatorname{Image} T$. Thus, we write $u = u + 0$ where we think of u as an element of $\operatorname{Image} T$ and 0 as an element of N. So $S(u) = v$ by definition of the map S. Therefore $ST(v) = v$ for each $v \in V$. Thus ST is the identity map from V to V.

So we have showed that if T is injective, then there is a linear map S s.t. ST is the identity.

Now suppose for $T : V \to W$ there is a map $S : W \to V$ s.t. $ST : V \to V$ is the identity map. We need to show that T is injective. So suppose that there are elements $v, v' \in V$ s.t. $T(v) = T(v')$. Applying S to both sides of this equality, we get that $ST(v) = ST(v')$. But ST is the identity, so $ST(v) = v$ and $ST(v') = v'$. Thus we get $v = v'$. Therefore $T(v) = T(v')$ implies $v = v'$. This means that T is injective.

□

Exercise 22 Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

Proof. Let $S, T \in \mathcal{L}(V)$. Suppose ST is invertible. We need to show that S and T are both invertible.

Since ST is invertible, there is a maps $R : V \to V$ s.t. $R(ST) = I$. Composition of maps is associative, so the first equation means

$$(RS)T = I$$

Since $\mathcal{L}(V) = \mathcal{L}(V, V)$ by definition, and since V is finite dimensional, the previous exercise (Exercise 14) implies that since there is a linear function, RS for which $(RS)T = I$, we must have that T is injective. By Theorem 3.21 in the book, since V is finite dimensional, T is injective iff it is invertible. Therefore T is invertible.

Since T is invertible, we can write $S = STT^{-1}$. Multiplying both sides of this equation by R on the left, we get $RS = T^{-1}$. Multiplying by T on the left, we get that $T(RS) = TT^{-1}$. So, $(TR)S = I$. Again, Exercise 14 implies that since we have a linear function, TR for which $(TR)S = I$, then S is injective. Then Theorem 3.21 implies that since S is injective, it is invertible.

Thus, if ST is invertible, so are S and T.

Suppose S, T are both invertible. Then we show that $(ST)^{-1} = T^{-1}S^{-1}$.

$$\begin{aligned}(T^{-1}S^{-1})ST &= T^{-1}(S^{-1}S)T \\ &= T^{-1}T \\ &= I\end{aligned}$$

and

$$\begin{aligned}ST(T^{-1}S^{-1}) &= S(TT^{-1})S^{-1} \\ &= S^{-1}S \\ &= I\end{aligned}$$

Thus $(ST)^{-1} = T^{-1}S^{-1}$, so ST is invertible. □

Exercise 23 Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ iff $TS = 1$.

Proof. Note that since S and T are arbitrary linear functions, we only need to show that $ST = I$ implies $TS = I$ (the other direction follows by switching the labels of our linear transformations). So suppose $ST = I$. By Exercise 14, $ST = I$ implies T is injective. Since V is finite dimensional, Theorem 3.21 implies T is invertible.

Since $ST = I$, we can multiply this equation by T on the left to get

$$TST = T$$

Multiplying both sides of this equation by T^{-1} on the right, we get

$$(TST)T^{-1} = I$$

Since function composition is associative, $(TST)T^{-1} = TS(TT^{-1})$. So we really have

$$TS = I$$

as required. □

Exercise 24 Suppose that V is finite dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity iff $ST = TS$ for every $S \in \mathcal{L}(V)$.

Proof. Suppose $T = aI$ for some $a \in \mathbb{F}$. Let $S \in \mathcal{L}(V)$. Then $S(aI)(v) = S(av) = aS(v)$ and $aI(S(v)) = aS(v)$. Thus $ST = TS$ for each $S \in \mathcal{L}(V)$.

Now suppose $ST = TS$ for each $S \in \mathcal{L}(V)$. Since V is finite dimensional, let $v_1, \dots, v_n$ be a basis for V. Define maps $S_{ij} : V \to V$ by

$$S_{ij}(a_1 v_1 + \cdots + a_n v_n) = a_i v_j$$

Clearly, this is a linear map. Now, for each v_i, let

$$T(v_i) = a_{i1} v_1 + a_{i2} v_2 + \cdots + a_{in} v_n$$

Then choosing numbers i and j between 1 and n,

$$\begin{aligned} S_{ij}T(v_i) &= a_{ii} v_j \text{ while} \\ TS_{ij}(v_i) &= T(v_j) \\ &= a_{j1} v_1 + \cdots + a_{jn} v_n \end{aligned}$$

So, since $S_{ij}T = TS_{ij}$,

$$a_{ii} v_j = a_{j1} v_1 + \cdots + a_{jn} v_n$$

Since $v_1, \dots, v_n$ form a basis, these two sums are equal iff the coefficients are equal. On the left hand side, only the coefficient on v_j is non-zero. On the right hand side, that coefficient is a_{jj}. Thus, $a_{jk} = 0$ for all $k \neq j$ and $a_{jj} = a_{ii}$

Since i and j were chosen arbitrarily, we get that $a_{ij} = 0$ for all $i \neq j$ and $a_{ii} = a_{jj}$ for all i and j. Let $a = a_{11}$ (this is also equal to a_{ii} for any i, since all the a_{ii} are equal). Then we showed that

$$\begin{aligned} T(v_i) &= a_{ii} v_i \\ &= a v_i \end{aligned}$$

for each i.

So, if $v = b_1 v_1 + \cdots + b_n v_n \in V$, then $Tv = b_1 T(v_1) + \cdots + b_n T(v_n)$. Thus,

$$\begin{aligned} Tv &= b_1(a v_1) + \cdots + b_n(a v_n) \\ &= a(b_1 v_1 + \cdots + b_n v_n) \\ &= av \end{aligned}$$

Thus, $T = aI$, so we are done. □

Question 1. Assume that $T \in \mathcal{L}(V)$. Recall that T^2 denotes the composition $T \circ T$.

a) Give an example of a vector space V and a linear operator $T \in \mathcal{L}(V)$ such that $T^2 = T$. (Not $T = 1$ or 0.)
b) Prove that if $T^2 = T$ then $V = \operatorname{Null} T \oplus \operatorname{Null}(T - I)$.
c) Prove that if $V = \operatorname{Null} T + \operatorname{Null}(T - I)$ then $T^2 = T$.
d) Give an example of a vector space V and a linear operator $T \in \mathcal{L}(V)$ such that $T^2 = -I$.

Proof.

a) First we give an example of a vector space V and a linear operator $T \in \mathcal{L}(V)$ such that $T^2 = T$. Let $V = \mathbb{R}^2$ and let T be the linear map given by

$$T(x, y) = (x + y, 0)$$

Then $T^2(x, y) = T(x + y, 0) = (x + y, 0)$. So $T(x, y) = T^2(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

b) Next we prove that if $T^2 = T$ then $V = \operatorname{Null} T \oplus \operatorname{Null}(T - I)$.

We have that $\operatorname{Null}(T - I) = \{v \mid (T - I)v = 0\}$. But $(T - I)(v) = 0$ iff $Tv = Iv$, that is, iff $Tv = v$. Thus,

$$\operatorname{Null}(T - I) = \{v \mid Tv = v\}$$

We want to show that $V = \operatorname{Null} T \oplus \operatorname{Null}(T - I)$. To do this, we first need to show that $V = \operatorname{Null} T + \operatorname{Null}(T - I)$. Let $v \in V$. We need to show that we can find $n \in \operatorname{Null} T$ and $u \in \operatorname{Null}(T - I)$ s.t. $v = n + u$. Since $T(Tv) = Tv$, we have that $Tv \in \operatorname{Null}(T - I)$.

Consider the vector $Tv - v$. Then

$$T(Tv - v) = T^2 v - Tv = 0$$

since $T^2 v = Tv$. Thus, for any v, $Tv - v \in \operatorname{Null} T$.

We can write $v = Tv - (Tv - v)$ where $Tv \in \operatorname{Null}(T - I)$ and $-(Tv - v) \in \operatorname{Null} T$. Thus $V = \operatorname{Null} T + \operatorname{Null}(T - I)$.

Next we need to show that $\operatorname{Null} T \cap \operatorname{Null}(T - I) = \{0\}$. Suppose $v \in \operatorname{Null} T \cap \operatorname{Null}(T - I)$. Then $v \in \operatorname{Null}(T - I)$ implies that $Tv = v$. On the other hand, $v \in \operatorname{Null} T$ implies that $Tv = 0$. Thus $v = 0$. Therefore $\operatorname{Null} T \cap \operatorname{Null}(T - I) = \{0\}$.

Since $\operatorname{Null} T \cap \operatorname{Null}(T - I) = \{0\}$ and $V = \operatorname{Null} T + \operatorname{Null}(T - I)$ we have shown that $V = \operatorname{Null} T \oplus \operatorname{Null}(T - I)$.

c) We want to show that if $V = \operatorname{Null} T + \operatorname{Null}(T - I)$ then $T^2 = T$. Let $v \in V$. Since $V = \operatorname{Null} T + \operatorname{Null}(T - I)$, we can find $n \in \operatorname{Null} T$ and $u \in \operatorname{Null}(T - I)$ s.t. $v = n + u$. Thus, we have

$$\begin{aligned} Tv &= T(n + u) \\ &= Tn + Tu \\ &= 0 + u \end{aligned}$$

So $Tv = u$, and since we showed above that $u \in \operatorname{Null}(T - I)$ implies $Tu = u$,

$$\begin{aligned} T^2 v &= Tu \\ &= u \end{aligned}$$

So $T^2 v = u$, as well. Thus $V = \operatorname{Null} T + \operatorname{Null}(T - I)$ implies $T^2 v = Tv$ for all $v \in V$.

d) Let $V = \mathbb{R}^2$ and let $T \in \mathcal{L}(V)$ be defined by $T(x, y) = (y, -x)$. Then $T^2(x, y) = T(y, -x) = (-x, -y)$. Thus $T^2(x, y) = -(x, y)$ for all $(x, y) \in \mathbb{R}^2$, so $T^2 = -I$. (Note that the linear operator T is just rotation by 90 degrees about the origin. Thus, squaring it, i.e. doing it twice, gives rotation by 180 degrees, which sends each vector to its opposite, negative, vector.)

$\square$

Question 2. Let V, W be finite dimensional, and consider $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$.

a) Prove that $\dim(\operatorname{Image} ST) \leq \dim(\operatorname{Image} T)$.
b) Prove that $\dim(\operatorname{Image} ST) = \dim(\operatorname{Image} T)$ if and only if

$$\operatorname{Image} T + \operatorname{Null} S = \operatorname{Image} T \oplus \operatorname{Null} S$$

c) Prove that $\dim(\operatorname{Null} ST) \leq \dim(\operatorname{Null} S) + \dim(\operatorname{Null} T)$.
d) Bonus: give a description (in terms of conditions on T, S, V, etc) of when $\dim(\operatorname{Null} ST) = \dim(\operatorname{Null} S) + \dim(\operatorname{Null} T)$.

[Proof by TC.] Before tackling these questions themselves we state and prove some lemmas, since we will use them in multiple parts. This will simplify our proofs.

First, we can restrict the domain of S to a obtain a map just from $\operatorname{Image} T$ to V,

$$S_{\operatorname{Im} T} : \operatorname{Image} T \to V$$

where $S_{\operatorname{Im} T}$ is defined to be the map from $\operatorname{Image} T$ to V defined by $S_{\operatorname{Im} T}(w) = S(w)$ for each $w \in \operatorname{Image} T$. Note that since S is linear, $S_{\operatorname{Im} T}$ must be linear as well.

Lemma 1. *$\operatorname{Image} S_{\operatorname{Im} T} = \operatorname{Image} ST$.*

Proof. Suppose that $v \in \operatorname{Image} ST$. Then there is some $v' \in V$ s.t. $v = STv'$. Since $Tv' \in \operatorname{Image} T$, we have that $S(Tv') = S_{\operatorname{Im} T}(Tv')$. Thus $v \in \operatorname{Image} ST$ implies $v \in \operatorname{Image} S_{\operatorname{Im} T}$.

Suppose $v \in \operatorname{Image} S_{\operatorname{Im} T}$. Then there is a $w \in \operatorname{Image} T$ s.t. $v = S(w)$. Since $w \in \operatorname{Image} T$, there is a $v' \in V$ s.t. $Tv' = w$. Thus $v = STv'$. So $v \in \operatorname{Image} S_{\operatorname{Im} T}$ implies $v \in \operatorname{Image} ST$. Thus $\operatorname{Image} ST = \operatorname{Image} S_{\operatorname{Im} T}$. □

Lemma 2. *$\operatorname{Null} S_{\operatorname{Im} T} = \operatorname{Image} T \cap \operatorname{Null} S$.*

Proof.

$$\begin{aligned}
\operatorname{Null} S_{\operatorname{Im} T} &= \left\{ w \in \operatorname{Image} T \middle| S_{\operatorname{Im} T}(w) = 0 \right\} \\
&= \left\{ w \in \operatorname{Image} T \middle| S(w) = 0 \right\} \\
&= \left\{ w \in W \middle| w \in \operatorname{Image} T \text{ and } S(w) = 0 \right\} \\
&= \left\{ w \in W \middle| w \in \operatorname{Image} T \right\} \cap \left\{ w \in W \middle| S(w) = 0 \right\} \\
&= \operatorname{Image} T \cap \operatorname{Null} S
\end{aligned}$$

□

Lemma 3. *$\dim \operatorname{Image} T = \dim(\operatorname{Image} T \cap \operatorname{Null} S) + \dim \operatorname{Image} ST$.*

Proof. Note that since W is finite dimensional, $\operatorname{Image} T$ is finite dimensional. Therefore the Rank-Nullity Theorem (Theorem 3.4 in the book) applied to $S_{\operatorname{Im} T}$ states:

$$\dim \operatorname{Image} T = \dim \operatorname{Null} S_{\operatorname{Im} T} + \dim \operatorname{Image} S_{\operatorname{Im} T}$$

Lemma 1 tells us that $\operatorname{Image} S_{\operatorname{Im} T} = \operatorname{Image} ST$, and Lemma 2 tells us that $\operatorname{Null} S_{\operatorname{Im} T} = \operatorname{Image} T \cap \operatorname{Null} S$. Substituting these, we obtain the desired equation. □

Lemma 4. *$\dim \operatorname{Null} ST = \dim \operatorname{Null} T + \dim(\operatorname{Image} T \cap \operatorname{Null} S)$*

Proof. By the Rank-Nullity Theorem applied to T we have

$$\dim V = \dim \operatorname{Null} T + \dim \operatorname{Image} T.$$

Using Lemma 3 to subsitute for $\dim \operatorname{Image} T$, this becomes

$$(*) \qquad \dim V = \dim \operatorname{Null} T + \dim(\operatorname{Image} T \cap \operatorname{Null} S) + \dim \operatorname{Image} ST$$

However, applying the Rank-Nullity Theorem to ST gives that

$$(**) \qquad \dim V = \dim \operatorname{Null} ST + \dim \operatorname{Image} ST$$

Subtracting $(**)$ from $(*)$ yields

$$0 = \dim \operatorname{Null} T + \dim(\operatorname{Image} T \cap \operatorname{Null} S) - \dim \operatorname{Null} ST$$

which becomes

$$\dim \operatorname{Null} ST = \dim \operatorname{Null} T + \dim(\operatorname{Image} T \cap \operatorname{Null} S)$$

as required. □

We now begin the proofs of Question 2(a)–(d).

a) We need to show that $\dim(\operatorname{Image} ST) \leq \dim(\operatorname{Image} T)$. By Lemma 3, we have

$$\dim \operatorname{Image} ST = \dim \operatorname{Image} T - \dim(\operatorname{Image} T \cap \operatorname{Null} S).$$

Since the dimension of $\operatorname{Image} T \cap \operatorname{Null} S$ must be ≥ 0, this implies $\dim \operatorname{Image} T \geq \dim \operatorname{Image} ST$ as required. [1] □

b) By Lemma 3, we have $\dim \operatorname{Image} ST = \dim \operatorname{Image} T - \dim(\operatorname{Image} T \cap \operatorname{Null} S)$. Therefore $\dim \operatorname{Image} ST = \dim \operatorname{Image} T$ if and only if $\dim(\operatorname{Image} T \cap \operatorname{Null} S) = 0$. Since the only 0-dimensional vector space is $\{0\}$, this holds if and only if $\operatorname{Image} T \cap \operatorname{Null} S = \{0\}$. But Proposition 1.9' says that for two subspaces U, W

$$U \cap W = \{0\} \iff U + W = U \oplus W.$$

Therefore applying Proposition 1.9' we have

$$\begin{aligned} \dim \operatorname{Image} ST = \dim \operatorname{Image} T &\iff \operatorname{Image} T \cap \operatorname{Null} S = \{0\} \\ &\iff \operatorname{Image} T + \operatorname{Null} S = \operatorname{Image} T \oplus \operatorname{Null} S, \end{aligned}$$

as required. [2] □

c) By Lemma 4 we know that $\dim \operatorname{Null} ST = \dim \operatorname{Null} T + \dim(\operatorname{Image} T \cap \operatorname{Null} S)$. Since $\operatorname{Image} T \cap \operatorname{Null} S$ is a subspace of $\operatorname{Null} S$, Prop. 2.15 states that $\dim(\operatorname{Image} T \cap \operatorname{Null} S) \leq \dim \operatorname{Null} S$. Therefore

$$\dim(\operatorname{Null} ST) = \dim \operatorname{Null} T + \dim(\operatorname{Image} T \cap \operatorname{Null} S) \leq \dim(\operatorname{Null} T) + \dim(\operatorname{Null} S)$$

as required. □

d) I claim that $\dim(\operatorname{Null} ST) = \dim(\operatorname{Null} T) + \dim(\operatorname{Null} S)$ if and only if **$\mathbf{Null\, S \subset Image\, T}$**.

By Lemma 4, $\dim(\operatorname{Null} ST) = \dim \operatorname{Null} T + \dim(\operatorname{Image} T \cap \operatorname{Null} S)$. This will be equal to $\dim(\operatorname{Null} T) + \dim(\operatorname{Null} S)$ if and only if $\dim(\operatorname{Image} T \cap \operatorname{Null} S) = \dim(\operatorname{Null} S)$. But $\operatorname{Image} T \cap \operatorname{Null} S$ is a subspace of $\operatorname{Null} S$, so by Exercise 2.11 on HW2 we know that

$$\dim(\operatorname{Image} T \cap \operatorname{Null} S) = \dim \operatorname{Null} S \iff \operatorname{Image} T \cap \operatorname{Null} S = \operatorname{Null} S.$$

But this last condition holds if and only if $\operatorname{Null} S \subset \operatorname{Image} T$, as claimed. [This is a general fact about sets: for any sets X and Y it's true that $X \cap Y = Y \iff Y \subset X$. The proof is quite straightforward. -TC] □

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[1] Alternate proof of 2a by TC, not using lemmas: By Rank-Nullity Theorem applied to ST, $\dim(\operatorname{Image} ST) = \dim V - \dim(\operatorname{Null} ST)$. By Rank Nullity Theorem applied to T, $\dim(\operatorname{Image} T) = \dim V - \dim(\operatorname{Null} T)$. We saw in class that $\operatorname{Null} T \subset \operatorname{Null} ST$, so by Prop. 2.15, $\dim(\operatorname{Null} T) \leq \dim(\operatorname{Null} ST)$. Therefore $-\dim(\operatorname{Null} ST) \leq -\dim(\operatorname{Null} T)$ [negating reverses inequalities]. Adding $\dim V$ to both sides gives the desired inequality:

$$\dim(\operatorname{Image} ST) = \dim V - \dim(\operatorname{Null} ST) \leq \dim V - \dim(\operatorname{Null} T) = \dim(\operatorname{Image} T).$$

[2] Alternate proof of 2b by TC (sketch): By the Rank-Nullity argument in previous footnote,

$$\dim(\operatorname{Image} ST) = \dim(\operatorname{Image} T) \iff \operatorname{Null} ST = \operatorname{Null} T.$$

This means $ST(v) = 0 \iff T(v) = 0$. Therefore $T(v) \neq 0 \implies ST(v) \neq 0$ (contrapositive of forwards implication). This means that if $w \in \operatorname{Image} T$ is nonzero, $S(w) \neq 0$; in other words, $\operatorname{Image} T \cap \ker S = \{0\}$. By Prop. 1.9, this is the condition to have a direct sum:

$$\operatorname{Image} T + \ker S = \operatorname{Image} T \oplus \ker S$$