1. Math 113 Homework 2 Solutions

Exercises from the book.

Exercise 1 Prove that if $(v_1, \ldots, v_n)$ spans V, then so does the list

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

Proof. Note that

$$v_1 = (v_1 - v_2) + (v_2 - v_3) + \cdots + (v_{n-1} - v_n) + v_n$$

and in general for $i < n$, that

$$v_i = (v_i - v_{i+1}) + (v_{i+1} - v_{i+2}) + \cdots + (v_{n-1} - v_n) + v_n$$

We want to show that for every $v \in V$ we can find elements $b_1, \ldots, b_n \in \mathbb{F}$ for which

$$v = b_1(v_1 - v_2) + \cdots + b_{n-1}(v_{n-1} - v_n) + b_n v_n$$

Since $(v_1, \ldots, v_n)$ spans V, then we can find elements $a_1, \ldots, a_n \in \mathbb{F}$ s.t.

$$v = a_1 v_1 + \cdots + a_n v_n$$

We can replace each v_i with the sums above to get

$$\begin{aligned} v =& a_1\big((v_1 - v_2) + (v_2 - v_3) + \cdots + (v_{n-1} - v_n) + v_n\big) \\ &+ a_2\big((v_2 - v_3) + (v_3 - v_4) + \cdots + (v_{n-1} - v_n) + v_n\big) \\ &\vdots \\ &+ a_{n-1}\big((v_{n-1} - v_n) + v_n\big) \\ &+ a_n v_n \end{aligned}$$

We see that $(v_1 - v_2)$ only appears once with coefficient a_1, $(v_2 - v_3)$ appears twice, with coefficients a_1 and a_2, and so on, until we get to v_n which appears n times with coefficients $a_1, \ldots, a_n$. So let $b_1 = a_1$, $b_2 = a_1 + a_2$, and more generally, $b_i = a_1 + \cdots + a_i$. Then

$$v = b_1(v_1 - v_2) + \cdots + b_{n-1}(v_{n-1} - v_n) + b_n v_n$$

So we have shown that $(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$ is a spanning set for V. □

Exercise 5 Prove that $\mathbb{F}^\infty$ is infinite dimensional.

Proof. Suppose for contradiction that $\mathbb{F}^\infty$ is finite dimensional, with dimension n. Now consider the vectors $v_1 = (1, 0, 0, \ldots)$, $v_2 = (0, 1, 0, \ldots)$, $v_3 = (0, 0, 1, \ldots)$ and so on until we get to the vector v_{n+1} with a 1 in the $(n+1)^{st}$ spot and zeros everywhere else.

If $\mathbb{F}^\infty$ has dimension n, then it has a basis with n elements. This basis is a spanning set. Theorem 2.6 from the book says that every linearly independent set is shorter than every spanning set. Thus, every linearly independent set has fewer than n elements. This means the $n + 1$ vectors $v_1, \ldots, v_{n+1}$ have to be linearly dependent. That is, we can find elements $a_1, \ldots, a_{n+1} \in \mathbb{F}$ not all of which were zero, s.t.

$$a_1 v_1 + \cdots + a_{n+1} v_{n+1} = 0$$

Now, the vector $a_1v_1 + \cdots + a_{n+1}v_{n+1}$ is just $(a_1, a_2, \dots, a_{n+1}, 0, 0, \dots)$. For this to be the zero vector, $a_1, \dots, a_{n+1}$ would have to all be zero. But this contradicts the fact that $v_1, \dots, v_{n+1}$ are linearly dependent.

Thus, $\mathbb{F}^\infty$ cannot have dimension n for any number n, so it is not finite dimensional. Therefore, $\mathbb{F}^\infty$ is infinite dimensional. □

Exercise 8 Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 = 3x_2, x_3 = 7x_4\}$$

Find a basis of U.

Proof. We will show that

$$\begin{aligned} v_1 &= (3, 1, 0, 0, 0) \\ v_2 &= (0, 0, 7, 1, 0) \\ v_3 &= (0, 0, 0, 0, 1) \end{aligned}$$

is a basis for U.

First, we'll show that (v_1, v_2, v_3) spans U. Let $v = (x_1, x_2, x_3, x_4, x_5) \in U$. Since $x_1 = 3x_2$ and $x_3 = 7x_4$ for all $u \in U$, we actually have $v = (3x_2, x_2, 7x_4, x_4, x_5)$. Then $v = x_2v_1 + x_4v_2 + x_5v_3$. Thus, we can write v as a linear combination of v_1, v_2 and v_3. Therefore, (v_1, v_2, v_3) spans U.

Next, we'll show that (v_1, v_2, v_3) is a linearly independent set. Suppose we have numbers $a_1, a_2, a_3 \in \mathbb{R}$ for which $a_1v_1 + a_2v_2 + a_3v_3 = 0 \in \mathbb{R}^5$. Expanding this out, we get

$$\begin{aligned} a_1v_1 + a_2v_2 + a_3v_3 &= 0 & \implies \\ (3a_1, a_1, 7a_2, a_2, a_3) &= (0, 0, 0, 0, 0) & \implies \\ a_1 = 0, a_2 = 0, a_3 &= 0 \end{aligned}$$

because two vectors in $\mathbb{R}^5$ are equal iff their coordinates are equal. Thus, v_1, v_2 and v_3 are linearly independent.

Since v_1, v_2 and v_3 span U and are linearly independent, they form a basis for U. (Note that since there are three of them, U has dimension 3.) □

Exercise 9 Prove or disprove: there exists a basis (p_0, p_1, p_2, p_3) of $P_3(\mathbb{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Proof. We will show that

$$\begin{aligned} p_0 &= 1 \\ p_1 &= x \\ p_2 &= x^3 + x^2 \\ p_3 &= x^3 \end{aligned}$$

is a basis for $P_3(\mathbb{F})$. Note that none of these polynomials has degree 2.

Proposition 2.16 in the book states that if V is a finite dimensional vector space, and we have a spanning list of vectors of length $\dim V$, then that list is a basis. It is shown in the book that $P_3(\mathbb{F})$ has dimension 4. Since this list has 4 vectors, we only need to show that it spans $P_3(\mathbb{F})$.

Suppose $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in P_3(\mathbb{F})$. We need to find $b_0, \ldots, b_3$ s.t. $p(x) = b_0p_0 + \cdots + b_3p_3$. Note that $p_2 - p_3 = x^2$. So let $b_0 = a_0, b_1 = a_1$, $b_2 = a_2$ and $b_3 = a_3 - a_2$. Then,

$$\begin{aligned} b_0p_0 + b_1p_1 + b_2p_2 + b_3p_3 &= a_0 + a_1x + a_2(x^2 + x^3) + (a_3 - a_2)x^3 \\ &= a_0 + a_1x + a_2x^2 + a_2x^3 + a_3x^3 - a_2x^3 \\ &= a_0 + a_1x + a_2x^2 + a_3x^3 \\ &= p(x) \end{aligned}$$

So we can can write $p(x)$ as a linear combination of p_0, p_1, p_2 and p_3. Thus p_0, p_1, p_2 and p_3 span $P_3(\mathbb{F})$. Thus, they form a basis for $P_3(\mathbb{F})$. Therefore, there exists a basis of $P_3(\mathbb{F})$ with no polynomial of degree 2. □

Exercise 11 Supose that V is finite dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove $U = V$.

Proof. Suppose $\dim U = \dim V = n$. Then we can find a basis $(u_1, \ldots, u_n)$ for U.

Since $(u_1, \ldots, u_n)$ is a basis of U, it is a linearly independent set. Proposition 2.17 says that if V is finite dimensional, then every linearly independent list of vectors in V of length $\dim V$ is a basis for V. The list $u_1, \ldots, u_n$ is a list of n linearly independent vectors in V (because it forms a basis for U, and because $U \subset V$.) Since $\dim V = n$, $(u_1, \ldots, u_n)$ is a basis of V.

This means that $(u_1, \ldots, u_n)$ spans V. Thus, we can express any $v \in V$ as a linear combination of $(u_1, \ldots, u_n)$. But each u_i is an element of U. Since U is a vector space, any linear combination of elements of U is also in U. Thus any $v \in V$ is also an element of U. Therefore $V \subset U$.

We have $U \subset V$ since U is a subspace of V, and we have just shown that $V \subset U$. Therefore, $U = V$. □

Exercise 14 Suppose U and W are both five-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

Proof. Suppose that $U \cap W = \{0\}$. By Theorem 2.18 in the book,

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

First consider the right hand side of this equation. Since $U \cap W = \{0\}$, the dimension of $U \cap W$ is zero. Since $\dim U = \dim V = 5$, the right hand side of this equation is 10.

Now consider the left hand side of the equation. The vector space $U + W$ is a subspace of $\mathbb{R}^9$. By Proposition 2.15 in the book, the dimension of a subspace of $\mathbb{R}^9$ is at most the dimension of $\mathbb{R}^9$. Since $\dim(R^9) = 9$, we have $\dim(U + W) \leq 9$. But this is impossible since the right hand side of the equality is 10.

Therefore, $U \cap W \neq \{0\}$. □

Exercise 16 Prove that if V is finite dimensional, and $U_1, \ldots, U_m$ are subspaces of V, then

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m$$

Proof. Let $V_i = U_1 + \cdots + U_i$. Then for each i, Theorem 2.18 tells us that

$$\dim(V_i + U_{i+1}) = \dim(V_i) + \dim(U_{i+1}) - \dim(V_i \cap U_{i+1})$$

In particular,

$$\dim(V_i + U_{i+1}) \leq \dim(V_i) + \dim(U_{i+1})$$

We will now apply this inequality repeatedly. Since $V_m = U_1 + \cdots + U_m$, we get:

$$\begin{aligned}
\dim(U_1 + \cdots + U_m) &\leq \dim(V_{m-1}) + \dim(U_m) \\
&\leq \dim(V_{m-2}) + \dim(U_{m-1}) + \dim(U_m) \\
&\vdots \\
&\leq \dim(V_2) + \dim(U_3) + \cdots + \dim U_m \\
&\leq \dim(U_1) + \dim(U_2) + \dim(U_3) + \cdots + \dim(U_m)
\end{aligned}$$

where the last inequality is true because $V_1 = U_1$. So, we are done. □

Exercise 17 Suppose V is finite dimensional. Prove that if $U_1, \ldots, U_m$ are subspaces of V such that $V = U_1 \oplus \cdots \oplus U_m$, then

$$\dim V = \dim U_1 + \cdots + \dim U_m$$

Proof. By Proposition 1.8 from the book, $V = U_1 \oplus \cdots \oplus U_m$ iff $V = U_1 + \cdots + U_m$ and that the only way to write 0 as a sum of $u_1, \ldots, u_n$ with $u_i \in U_i$ is to choose $u_i = 0$ for each i.

Let $V_i = U_1 + \cdots + U_i$. We claim that in fact, $V_i = U_1 \oplus \cdots \oplus U_i$. If this were not true, we could write zero as a sum $u_1 + \cdots + u_i = 0$ for $u_j \in U_j$ and some $u_j \neq 0$. But then $0 = u_1 + u_2 + \cdots + u_i + 0 + \cdots + 0$, where we think of the zeros as elements of $u_{i+1}, \ldots, u_m$. So we can write 0 as a sum of elements of $U_1, \ldots, U_m$, not all of which are 0. Then Proposition 1.8 implies that V is not the direct sum of $U_1, \ldots, U_m$. This is a contradiction. Thus, $V_i = U_1 \oplus \cdots \oplus U_i$.

Theorem 2.18 from the book that says that for subspaces V_i, U_{i+1} of V,

$$\dim(V_i + U_{i+1}) = \dim V_i + \dim U_{i+1} - \dim(V_i \cap U_{i+1})$$

We will show that $V_i \cap U_{i+1} = \{0\}$. Suppose $v \in V_i \cap U_{i+1}$ with $v \neq 0$. Since $v \in V_i$, we can write $v = u_1 + \cdots + u_i$ for $u_j \in U_j$. Since $v \in U_{i+1}$, we can also write $v = u_{i+1} \in U_i$. Since $0 = v - v$, we can write $0 = u_1 + \cdots + u_i - u_{i+1}$ for $u_j \in U_j$. But since $V_{i+1} = V_i + U_{i+1} = U_1 + \cdots + U_{i+1}$ is actually the direct sum of $U_1, \ldots, U_{i+1}$, this is impossible by Proposition 1.8. Therefore, $V_i \cap U_{i+1} = \{0\}$.

Thus, $\dim(V_i \cap U_{i+1}) = 0$. So, by Theorem 2.18,

$$\dim(V_i + U_{i+1}) = \dim V_i + \dim U_{i+1}$$

for each i. Just like for exercise 16, we will now use this formula multiple times. The first equality in what follows comes from the fact that $V_m = U_1 + \cdots + U_m$.

$$\begin{aligned}
\dim(U_1 + \cdots + U_m) &= \dim(V_{m-1}) + \dim(U_m) \\
&= \dim(V_{m-2}) + \dim(U_{m-1}) + \dim(U_m) \\
&\vdots \\
&= \dim(V_2) + \dim(U_3) + \cdots + \dim U_m \\
&= \dim(U_1) + \dim(U_2) + \dim(U_3) + \cdots + \dim(U_m)
\end{aligned}$$

where the last equality comes from the fact that $V_1 = U_1$. Thus, $\dim(U_1 + \cdots + U_m) = \dim(U_1) + \dim(U_2) + \cdots + \dim(U_m)$. Since $V = U_1 + \cdots + U_m$ we are done. □

Question 1. If V is a vector space over the field $\mathbb{F}$, consider its dual vector space V^*.

- By definition, V^* is a subset of the vector space $\mathbb{F}^V$ of functions from V to $\mathbb{F}$. Prove that V^* is a subspace of $\mathbb{F}^V$.
- Assume that $\dim V = n$, and that $(v_1, \ldots, v_n)$ is a basis for V. Find a basis V^*. What is $\dim V^*$?

Proof.

- First we show that V^* is a subspace of $\mathbb{F}^V$. We need to show the following things.

  Zero: We need to show that the zero function $\mathbb{O} : V \to \mathbb{F}$ with $\mathbb{O}(v) = 0$ for all $v \in V$ satisfies the conditions for an element of V^*. For $v, w \in V$, and for $a \in \mathbb{F}$,

$$\begin{aligned} \mathbb{O}(v+w) &= 0 \\ &= \mathbb{O}(v) + \mathbb{O}(w) \end{aligned}$$

  and

$$\begin{aligned} \mathbb{O}(av) &= 0 \\ &= a\mathbb{O}(v) \end{aligned}$$

  since $\mathbb{O}$ sends every element of V to zero. Thus, the zero function is in V^*.

  Closed Under Vector Addition: We need to show that if $f, g \in V^*$ then $f + g \in V^*$. Again, we need to show that $f + g$ satisfies the condition of an element of V^*. For $v, w \in V$, and for $a \in \mathbb{F}$,

$$\begin{aligned} (f+g)(v+w) &= f(v+w) + g(v+w) \\ &= (f(v) + f(w)) + (g(v) + g(w)) \text{ because } f, g \in V^* \\ &= (f+g)(v) + (f+g)(w) \end{aligned}$$

  and

$$\begin{aligned} (f+g)(av) &= f(av) + g(av) \\ &= af(v) + ag(v) \text{ because } f, g \in V^* \\ &= a(f+g)(v) \end{aligned}$$

  Therefore, $f + g \in V^*$, so V^* is closed under vector addition.

  Closed Under Scalar Multiplication: Given a scalar $b \in \mathbb{F}$ and a function $f \in V^*$, we need to show that $bf \in V^*$. For $v, w \in V$, and for $a \in \mathbb{F}$,

$$\begin{aligned} bf(v+w) &= b(f(v) + f(w)) \text{ because } f \in V^* \\ &= bf(v) + bf(w) \end{aligned}$$

  and

$$\begin{aligned} bf(av) &= b(af(v)) \text{ because } f \in V^* \\ &= a(bf(v)) \end{aligned}$$

  Therefore, V^* is closed under scalar multiplication.

  Thus, V^* is a vector space.

- Let $(v_1, \ldots, v_n)$ be a basis for V. Let $v \in V$. Then we can write v as a linear combination of the basis vectors. That is, $v = a_1 v_1 + \cdots + a_n v_n$ for some $a_1, \ldots, a_n \in \mathbb{F}$. Let $f_i : V \to \mathbb{F}$ be the map such that $f_i(v) = a_i$. We will show that $(f_1, \ldots, f_n)$ is a basis for V^*.

  First, we should show that $f_i \in V^*$ for all i. Suppose $v, w \in V$. Write $v = a_1 v_1 + \cdots + a_n v_n$ and $w = b_1 v_1 + \cdots + b_n v_n$. Thus $f_i(v) = a_i$ and $f_i(w) = b_i$. We have $v + w = (a_1 + b_1)v_1 + \ldots (a_n + b_n)v_n$. So, $f_i(v + w) = a_i + b_i$. This is the same as $f_i(v) + f_i(w)$. Thus $f_i(v + w) = f_i(v) + f_i(w)$. Next, if $c \in \mathbb{F}$, then $cv = ca_1 v_1 + \cdots + ca_n v_n$. So, $f_i(cv) = ca_i$, which is the same as $cf_i(v)$. Thus $f_i(cv) = cf_i(v)$. Therefore, $f_i \in V^*$ for all i.

  Next, we need to show that $f_1, \ldots, f_n$ span V^*. Let $f \in V^*$. We will show that if $c_i = f(v_i)$, (where $(v_1, \ldots, v_n)$ is our basis for V), then $f = c_1 f_1 + \cdots + c_n f_n$.

  Let $v = a_1 v_1 + \cdots + a_n v_n \in V$. Then

$$
\begin{aligned}
f(v) &= a_1 f(v_1) + \ldots a_n f(v_n) \text{ because } f \in V^* \\
&= a_1 c_1 + \cdots + a_n c_n \text{ since we defined } c_i = f(v_i) \\
&= c_1 f_1(v) + \cdots + c_n f_n(v) \text{ because } f_i(v) = a_i, \text{ by definition}
\end{aligned}
$$

  Thus $f(v) = (c_1 f_1 + \ldots c_n f_n)(v)$ for all $v \in V$, so $f = c_1 f_1 + \ldots c_n f_n$. This shows that any element f of V^* can be written as a linear combination of $f_1, \ldots, f_n$.

  Lastly, we need to show that $f_1, \ldots, f_n$ are linearly independent. Suppose that we can find constants $c_1, \ldots, c_n$ s.t. $c_1 f_1 + \cdots + c_n f_n = 0$. Then for any element $v \in V$, $c_1 f_1(v) + \cdots + c_n f_n(v) = 0$. In particular, for any i,

$$
\begin{aligned}
c_1 f_1(v_i) + \cdots + c_n f_n(v_i) = 0 &\text{ but } f_j(v_i) = 0 \text{ for all } j \neq i \text{ so} \\
c_i f_i(v_i) = 0 &\text{ and since } f_i(v_i) = 1. \\
c_i = 0&
\end{aligned}
$$

  Therefore $c_i = 0$ for all i. Thus the f_i are linearly independent.

  This means $f_1, \ldots, f_n$ form a basis for V^*. Since there are n elements of this basis, the dimension of V^* is n.

  □

Question 2. We define $\mathcal{L}(V, W)$ to be the subset of W^V consisting of linear functions $T : V \to W$.

- Prove the $\mathcal{L}(V, W)$ is a subspace of W^V.
- Assume that $\dim V = 3$ and $\dim W = 2$, and furthermore assume that (v_1, v_2, v_3) is a basis for V and (w_1, w_2) is a basis for W. Find a basis for $\mathcal{L}(V, W)$. What is $\dim \mathcal{L}(V, W)$.

Proof.
- First we show that $\mathcal{L}(V, W)$ is a subspace of W^V.

  Zero: We need to show that the zero map $\mathbb{O} : V \to W$ is an element of $\mathcal{L}(V, W)$. Recall that $\mathbb{O}(v) = 0 \in W$ for all $v \in V$. So we need to show that $\mathbb{O}$ satisfies the two properties of a linear map. Let $v, w \in V$ and

$a \in \mathbb{F}$. Then

$$\begin{aligned}\mathbb{O}(v+w) &= 0 \\ &= \mathbb{O}(v) + \mathbb{O}(w) \text{ and} \\ \mathbb{O}(av) &= 0 \\ &= a\mathbb{O}(v)\end{aligned}$$

since $\mathbb{O}$ sends all vectors to the zero vector. Thus $\mathbb{O} \in \mathcal{L}(V, W)$.

Closed Under Vector Addition: Let $f, g \in \mathcal{L}(V, W)$. We need to show that $f + g$ is also a linear map from V to W. Let $v, w \in V$ and $a \in \mathbb{F}$.

$$\begin{aligned}(f+g)(v+w) &= f(v+w) + g(v+w) \\ &= (f(v) + f(w)) + (g(v) + g(w)) \text{ because } f, g \text{ are linear maps} \\ &= (f+g)(v) + (f+g)(w)\end{aligned}$$

and

$$\begin{aligned}(f+g)(av) &= f(av) + g(av) \\ &= af(v) + ag(v) \text{ because } f, g \text{ are linear maps} \\ &= a(f+g)(v)\end{aligned}$$

Closed Under Scalar Multiplication: Let $f \in \mathcal{L}(V, W)$ and $b \in \mathbb{F}$. We need to show that bf is also a linear map from V to W. Let $v, w \in V$ and $a \in \mathbb{F}$.

$$\begin{aligned}bf(v+w) &= b(f(v) + f(w)) \text{ because } f \text{ is a linear map} \\ &= bf(v) + bf(w)\end{aligned}$$

and

$$\begin{aligned}bf(av) &= b(af(v)) \text{ because } f \text{ is a linear map} \\ &= a(bf(v))\end{aligned}$$

Therefore $\mathcal{L}(V, W)$ is closed under scalar multiplication.

Thus, $\mathcal{L}(V, W)$ is a subspace of W^V.

- We assume that $\dim V = 3$ and $\dim W = 2$, and furthermore that (v_1, v_2, v_3) is a basis for V and (w_1, w_2) is a basis for W. Let $v \in V$. Then we can write $v = a_1v_1 + a_2v_2 + a_3v_3$. Define $f_{ij} : V \to W$ for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$ to be

$$f_{ij}(v) = a_i w_j$$

For example, $f_{12}(v) = a_1 w_2$, and so on.

We claim that $(f_{11}, f_{12}, f_{21}, f_{22}, f_{31}, f_{32})$ form a basis for $\mathcal{L}(V, W)$. First we need to show that f_{ij} is linear for any i, j. Suppose $v, w \in V$. Write $v = a_1v_1 + a_2v_2 + a_3v_3$ and $w = b_1v_1 + b_2v_1 + b_3v_3$. Thus $f_{ij}(v) = a_i w_j$ and $f_{ij}(w) = b_i w_j$. We have $v + w = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + (a_3 + b_3)v_3$. So, $f_{ij}(v + w) = (a_i + b_i)w_j$. This is the same as $f_{ij}(v) + f_{ij}(w)$. Thus $f_{ij}(v + w) = f_{ij}(v) + f_{ij}(w)$. If $c \in \mathbb{F}$, then $cv = ca_1v_1 + ca_2v_2 + ca_2v_3$. So, $f_{ij}(cv) = ca_i w_j$, which is the same as $cf_{ij}(v)$. Thus $f_{ij}(cw) = cf_{ij}(w)$. Therefore, $f_{ij} \in \mathcal{L}(V, W)$ for all i, j.

Next, we need to show that $(f_{11}, f_{12}, f_{21}, f_{22}, f_{31}, f_{32})$ span $\mathcal{L}(V, W)$. Let $f \in \mathcal{L}(V, W)$. For each i, $f(v_i)$ is an element of W. That means we can express it as a linear combination of w_1 and w_2. For each $i \in \{1, 2, 3\}$ we define elements $c_{i1}, c_{i2} \in \mathbb{F}$ s.t. $f(v_i) = c_{i1}w_1 + c_{i2}w_2$ (where (v_1, v_2, v_3) is our basis for V.) We claim that

$$f = c_{11}f_{11} + c_{12}f_{12} + c_{21}f_{21} + c_{22}f_{22} + c_{31}f_{31} + c_{32}f_{32}$$

Let $v = a_1v_1 + a_2v_2 + a_3v_3 \in V$. . Then

$$f(v) = a_1f(v_1) + a_2f(v_2) + a_3f(v_3)$$

because f is linear

$$= a_1(c_{11}w_1 + c_{12}w_2) + a_2(c_{21}w_1 + c_{22}w_2) + a_3(c_{31}w_1 + c_{32}w_2)$$

by the definition of c_{ij} for each i, j

$$= c_{11}f_{11}(v) + c_{12}f_{12}(v) + c_{21}f_{21}(v) + c_{22}f_{22}(v) + c_{31}f_{31}(v) + c_{32}f_{32}(v)$$

because $f_{ij}(v) = a_iw_j$, by definition

Thus any element of $\mathcal{L}(V, W)$ can be written as a linear combination of $f_{11}, f_{12}, f_{21}, f_{22}, f_{31}$ and f_{32}.

Lastly, we need to show that $f_{11}, f_{12}, f_{21}, f_{22}, f_{31}$ and f_{32} are linearly independent. Suppose that we can find constants $c_{11}, c_{12}, c_{21}, c_{22}, c_{31}$ and c_{32} s.t.

$$c_{11}f_{11} + c_{12}f_{12} + c_{21}f_{21} + c_{22}f_{22} + c_{31}f_{31} + c_{32}f_{32} = 0$$

Then for any element $v \in V$,

$$c_{11}f_{11}(v) + c_{12}f_{12}(v) + c_{21}f_{21}(v) + c_{22}f_{22}(v) + c_{31}f_{31}(v) + c_{32}f_{32}(v) = 0$$

In particular, for any i,

$$c_{11}f_{11}(v_i) + c_{12}f_{12}(v_i) + c_{21}f_{21}(v_i) + c_{22}f_{22}(v_i) + c_{31}f_{31}(v_i) + c_{32}f_{32}(v_i) = 0$$

but $f_{kj}(v_i) = 0$ for all $k \neq i$ so

$$c_{i1}f_{i1}(v_i) + c_{i2}f_{i2}(v_i) = 0$$

and since $f_{ij}(v_i) = w_j$,

$$c_{i1}w_1 + c_{i2}w_2 = 0$$

but w_1, w_2 form a basis for W, so this is only possible if

$$c_{i1} = c_{i2} = 0$$

Therefore $c_{ij} = 0$ for all i, j. Thus the f_{ij} are linearly independent.

This means $f_{11}, f_{12}, f_{21}, f_{22}, f_{31}$ and f_{32} form a basis for $\mathcal{L}(V, W)$. Since there are 6 elements of this basis, the dimension of $\mathcal{L}(V, W)$ is 6.

□

Question 3. Let U be a subset of $\mathbb{R}^\infty$ consisting of all sequences that satisfy

$$v_i + v_{i+2} = v_{i+1} \text{ for all } i$$

(1) Prove that U is a subspace of $\mathbb{R}^\infty$.
(2) Let $x, y \in U$ be the elements

$$x = (0, 1, 1, 0, -1, -1, 0, 1, 1, \dots)$$
$$y = (1, 0, -1, -1, 0, 1, 1, 0, -1, \dots)$$

Prove that (x, y) is a linearly independent set.
(3) Prove that (x, y) is a basis for U.
(4) Let W be the subspace of $\mathbb{R}^\infty$ consisting of all sequences with $v_1 = 0$ and $v_2 = 0$. Prove that $\mathbb{R}^\infty = U \oplus W$.

Proof. (1) First we prove that U is a subspace of $\mathbb{R}^\infty$. To do this, we show that it has the following properties.

Zero: The sequence $(0, 0, \dots)$ satisfies $v_i + v_{i+2} = v_{i+1}$ because $v_i = v_{i+1} = v_{i+2} = 0$. Therefore $0 \in U$.

Closed Under Vector Addition: Suppose $v = (v_1, v_2, \dots), w = (w_1, w_2, \dots) \in U$. Then $v_i + v_{i+2} = v_{i+1}$ and $w_i + w_{i+2} = w_{i+1}$. Thus $(v_i + w_i) + (v_{i+2} + w_{i+2}) = (v_{i+1} + w_{i+1})$. Since the i^{th} term of $v + w$ is $v_i + w_i$ for each i, this means that $v + w \in U$. Therefore U is closed under vector addition.

Closed Under Scalar Multiplication: Suppose $v = (v_1, v_2, \dots) \in U$ and $a \in \mathbb{R}$. Since $v_i + v_{i+2} = v_{i+1}$, we have that $av_i + av_{i+2} = av_{i+1}$. Since the i^{th} term of av is av_i for each i, this means that $av \in U$. Therefore U is closed under scalar multiplication.

Since U satisfies these properties, it is a subspace of $\mathbb{R}^\infty$.

(2) Let $x, y \in U$ be the elements

$$x = (0, 1, 1, 0, -1, -1, 0, 1, 1, \dots)$$
$$y = (1, 0, -1, -1, 0, 1, 1, 0, -1, \dots)$$

We will show that (x, y) is a linearly independent set.
Suppose not. Then we can find $a, b \in \mathbb{R}$ s.t. $ax + by = 0$. Note that

$$ax = (0, a, a, 0, -a, -a, 0, a, a, \dots)$$
$$by = (b, 0, -b, -b, 0, b, b, 0, -b, \dots) \text{ so,}$$
$$ax + by = (b, a, \dots)$$

If $ax + by = 0$ then $b = 0$ and $a = 0$ since two sequences are equal iff their terms are all equal. This means that x and y are linearly independent.

(3) Next we show that (x, y) is a basis for U. Since we have already shown that (x, y) is a linearly independent set, we just need to show that it spans U.

Let $u \in U$. Write $u = (u_1, u_2, \dots)$. Then we claim that $u = u_1 y + u_2 x$. Note that

$$u_1 y + u_2 x = (u_1, u_2, u_2 - u_1, -u_1, -u_2, -u_2 + u_1, u_1, u_2, u_2 - u_1, \dots)$$

We will show that all the terms of u and $u_1 y + u_2 x$ match up by induction. We will use the fact that since $u_i + u_{i+2} = u_{i+1}$, then $u_{i+2} = u_{i+1} - u_i$. First of all, this means that $u_3 = u_2 - u_1$. Thus, u and $u_1 y + u_2 x$ match up on the first three terms.

Now suppose the first $3n$ terms of u and $u_1 y + u_2 x$ are the same. We need to show that this implies the first $3(n+1)$ terms are the same. There are two cases: n is either odd or even. First suppose n is odd. Then

$$u = (u_1, \dots, -u_2 + u_1, u_1, u_2, u_2 - u_1, u_{3n+1}, u_{3n+2}, u_{3n+3}, \dots)$$

where the $u_2 - u_1$ is its $3n^{th}$ term. Since $u_{3n+1} = u_{3n} - u_{3n-1}$, we have that $u_{3n+1} = -u_1$. Next, since $u_{3n+2} = u_{3n+1} - u_{3n}$, we have that $u_{3n+2} = -u_2$. Lastly, since $u_{3n+3} = u_{3n+2} - u_{3n+1}$, we have that $u_{3n+3} = -u_2 + u_1$. Thus u and $u_1 y + u_2 x$ match up for $3n + 3 = 3(n + 1)$ terms.

Now suppose that n is even. Then,

$$u = (u_1, \dots, u_2 - u_1, -u_1, -u_2, -u_2 + u_1, u_{3n+1}, u_{3n+2}, u_{3n+3}, \dots)$$

where the $u_2 - u_1$ is its $3n^{th}$ term. Since $u_{3n+1} = u_{3n} - u_{3n-1}$, we have that $u_{3n+1} = u_1$. Next, since $u_{3n+2} = u_{3n+1} - u_{3n}$, we have that $u_{3n+2} = u_2$. Lastly, since $u_{3n+3} = u_{3n+2} - u_{3n+1}$, we have that $u_{3n+3} = u_2 - u_1$. Thus u and $u_1 y + u_2 x$ match up for $3n + 3 = 3(n + 1)$ terms.

Therefore, by induction, $u_1 y + u_2 x = u$.

This means that x and y span U. Since we have shown that they are linearly independent, they form a basis for U.

(4) Let W be the subspace of $\mathbb{R}^\infty$ consisting of all sequences with $v_1 = 0$ and $v_2 = 0$. We need to show that $\mathbb{R}^\infty = U \oplus W$. By Proposition 1.9 from the book, $\mathbb{R}^\infty = U \oplus W$ iff $\mathbb{R}^\infty = U + W$ and $U \cap W = \{0\}$.

To show that $\mathbb{R}^\infty = U + W$ we need to show that any sequence can be written as the sum of an element of U and an element of W. Let $x = (x_1, x_2, \dots) \in \mathbb{R}^\infty$. Let $u = (x_1, x_2, x_2 - x_1, -x_1, -x_2, x_1 - x_2, x_1, x_2, \dots)$ be the element of U that starts with x_1 and x_2. Let $w = x - u$. Since u and x have the same first and second term, $w = (0, 0, w_3, w_4, \dots)$. So, $w \in W$. Since $x = u + w$, we can write any element of $\mathbb{R}^\infty$ as the sum of an element of U plus an element of W. Thus, $\mathbb{R}^\infty = U + W$.

To show that $U \cap W = \{0\}$, suppose $v \in U \cap W$. We will show that $v = 0$ by induction. Write $v = (v_1, v_2, \dots)$. Since $v \in W$, $v_1 = v_2 = 0$. Suppose $v_{n-1} = v_n = 0$. Then we need to show that $v_{n+1} = 0$. Since $v_{n+1} = v_n - v_{n-1}$, we have that $v_{n+1} = 0$. So by induction, $v = 0$. Therefore, $U \cap W = \{0\}$.

Since we proved $\mathbb{R}^\infty = U + W$ and $U \cap W = \{0\}$, we have shown that $\mathbb{R}^\infty = U \oplus W$.

□