# Math 113 Homework 1 Solutions

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Exercises 3, 9, 14 and 15 are from the book.

**Exercise 3** Prove that $-(-v) = v$ for every $v \in V$.

*Proof.* The defining property of the additive inverse $-v$ is that $v + (-v) = 0$. We can add $-(-v)$ to both sides of this equation and get

$$[v + (-v)] + [-(-v)] = 0 + [-(-v)]$$

so by the associative and additive identity properties of vector spaces,

$$v + [(-v) + [-(-v)]] = [-(-v)]$$

Now since $-(-v)$ is the additive inverse of $(-v)$, we know that $(-v) + [-(-v)] = 0$, so this equation becomes

$$v + 0 = [-(-v)].$$

By the additive identity property, this becomes

$$v = [-(-v)]$$

which is what we set out to show. □

**Exercise 9** Prove that the union of two subspaces of $V$ is a subspace of $V$ if and only if one of the subspaces is contained in the other.

*Proof.* Let $U, W$ be subspaces of $V$, and let $V' = U \cup W$. First we show that if $V'$ is a subspace of $V$ then either $U \subset W$ or $W \subset U$. So suppose for contradiction that $V' = U \cup W$ is a subspace but neither $U \subset W$ nor $W \subset U$. Then we can choose some $u \in U \setminus W$ and some $w \in W \setminus U$. [1] That is, $u$ is in $U$ but not in $W$ and $w$ is in $W$ but not in $U$.

Since $V' = U \cup W$ and $u \in U$, we know that $u$ is in $V'$. Similarly, since $w \in W$, we know that $w$ is in $V'$. And since $V'$ is a subspace of $V$, $u + w$ must be in $V'$. All elements of $V'$ belong either to $U$ or to $W$. So either $u + w \in U$ or $u + w \in W$. We handle these two cases separately. First, suppose $u + w = u' \in U$. Then $w = u' - u$. Since $u'$ and $u$ are in $U$, and $U$ is a subspace, we have that $u' - u \in U$. Thus $w \in U$. Second (we use the same method), suppose that $u + w = w' \in W$. Then $u = w' - w$. Since $w'$ and $w$ are in $W$, and $W$ is a subspace, we have that $w' - w \in W$, and so $u \in W$. So in the first case we find that $w \in U$, and in the second case we find that $u \in W$. But we specifically chose $u$ and $w$ so that $u$ was not in $W$ and $w$ was not in $U$, so neither of these cases is possible. This gives us a contradiction.

Therefore, it was impossible to both choose some $u \in U \setminus W$ and some $w \in W \setminus U$. That is, either $U \setminus W$ is empty or $W \setminus U$ is empty. In the first case, $U \subset W$, and in the second case, $W \subset U$.

Next we show that if $U \subset W$ or $W \subset U$, then $V' = U \cup W$ is a subspace of $V$. But if $U \subset W$ then $U \cup W = W$, and $W$ is a subspace. And if $W \subset U$ then $U \cup W = U$, and $U$ is a subspace. So in either case, $V'$ is a subspace. Therefore,

[1] If $X$ and $Y$ are two sets, the *set difference* $X \setminus Y$ is the set of all elements that *are* in $X$ but are *not* in $Y$.

we have shown that the union of two subspaces of $V$ is a subspace of $V$ if and only if one of the subspaces is contained in the other. □

**Exercise 14** Suppose $U$ is the subspace of $P(\mathbb{F})$ consisting of all polynomials $p$ of the form

$$p(z) = az^2 + bz^5$$

where $a, b \in \mathbb{F}$. Find a subspace $W$ of $P(\mathbb{F})$ such that $P(\mathbb{F}) = U \oplus W$

*Proof.* Let $W$ be the subspace of $P(\mathbb{F})$ consisting of all polynomials of the form $a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$ where $a_2 = a_5 = 0$. This is a subspace: the zero polynomial is in it because in the zero polynomial, the coefficients of $z^2$ and $z^5$ are both zero. Suppose $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$ and $q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m$ are in $W$. The coefficients of $z^2$ and $z^5$ in $p(z) + q(z)$ are $a_2 + b_2$ and $a_5 + b_5$. Since $p(z), q(z) \in W$, $a_2 = b_2 = a_5 = b_5 = 0$. Thus the coefficients of $z^2$ and $z^5$ in $p(z) + q(z)$ are also zero, so $p(z) + q(z) \in W$. Lastly, if $c \in \mathbb{F}$, then the coefficients of $z^2$ and $z^5$ in $cp(z)$ are $ca_2$ and $ca_5$, respectively. Since $a_2 = a_5 = 0$, we have that the coefficients of $z^2$ and $z^5$ in $cp(z)$ are both zero. Thus $cp(z) \in W$. Therefore $W$ is a subspace.

We claim that $P(\mathbb{F}) = U \oplus W$. To show this, we will use Proposition 1.9 from the book that says $P(\mathbb{F}) = U \oplus W$ if and only if $P(\mathbb{F}) = U + W$ and $U \cap W = \{0\}$.

So first we show that $P(\mathbb{F}) = U + W$. That is, we want to show that given a polynomial $p(z) \in P(\mathbb{F})$, we can write $p(z) = q(z) + r(z)$ where $q(z) \in U$ and $r(z) \in W$. Suppose $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$. Then let $q(z) = a_2 z^2 + a_5 z^5$ and let $r(z) = p(z) - q(z)$. Then clearly $q(z) \in U$. And, since the coefficients of $z^2$ and $z^5$ of $r(z)$ are $a_2 - a_2 = 0$ and $a_5 - a_5 = 0$ respectively, $r(z) \in W$. We chose $q(z)$ and $r(z)$ so that $p(z) = q(z) + r(z)$, so we have shown that any polynomial can be written as a sum of a polynomial in $U$ plus a polynomial in $W$.

Next we show that $U \cap W = \{0\}$. Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$. Suppose $p(z) \in U \cap W$. That means $p(z) \in U$ so $a_i = 0$ for all $i \neq 2, 5$. It also means that $p(z) \in W$, so $a_2 = a_5 = 0$. Thus all of the coefficients of $p(z)$ are zero. So if $p(z) \in U \cap W$, $p(z)$ can only be the zero polynomial. Therefore $U \cap W = \{0\}$.

So we have shown that $P(\mathbb{F}) = U \oplus W$. □

**Exercise 15** Prove or give a counterexample: if $U_1, U_2, W$ are subspaces of $V$ such that

$$V = U_1 \oplus W \text{ and } V = U_2 \oplus W$$

then $U_1 = U_2$.

*Counterexample.* Let $V = \mathbb{R}^2$. Let $W$ be the $x$-axis. That is,

$$W = \{(x, 0) \mid x \in \mathbb{R}\}$$

This is a subspace: If we set $x = 0$, we see that $(0, 0) \in W$. And if we take $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$ we once again get a vector in $W$. Likewise, $r(x, 0) = (rx, 0) \in W$.

Let $U_1$ be the $y$-axis. That is,

$$U_1 = \{(0, y) \mid y \in \mathbb{R}\}$$

And finally, let $U_2$ be the line through the origin where $x = y$, so

$$U_2 = \{(x, x) \mid x \in \mathbb{R}\}$$

We see that these are all subspaces of $\mathbb{R}^2$ using calculations similar to those for $W$.

We claim that $\mathbb{R}^2 = U_1 \oplus W$ and $\mathbb{R}^2 = U_2 \oplus W$. We'll show this by using the result of Prop. 1.9 from the book, which says we only need to show that $\mathbb{R}^2$ is the sum of these subspaces, and that the intersection of these subspaces is just $\{0\}$, where in this case, the zero vector is $(0, 0)$.

Let $(x, y) \in \mathbb{R}^2$. Then we can write

$$(x, y) = (0, y) + (x, 0).$$

Since $(0, y) \in U_1$ and $(x, 0) \in W$, this shows that any $(x, y) \in \mathbb{R}^2$ can be written as a sum of elements of $U_1$ and $W$. Thus $\mathbb{R}^2 = U_1 + W$. Similarly, for any $(x, y) \in \mathbb{R}^2$ we can write

$$(x, y) = (y, y) + (x - y, 0).$$

Since $(y, y) \in U_2$ and $(x-y, 0) \in W$, this shows that any $(x, y) \in \mathbb{R}^2$ can be written as a sum of elements of $U_2$ and $W$. Thus $\mathbb{R}^2 = U_2 + W$.

We also need to show that $U_1 \cap W = \{(0, 0)\}$ and $U_2 \cap W = \{(0, 0)\}$. But we can see this geometrically. The $x$- and $y$- axes intersect only at the origin of $\mathbb{R}^2$, and the same is true for the $x$-axis and the line through the origin where $x = y$. Algebraically, we prove that $U_1 \cap W = \{(0, 0)\}$ as follows: if an element $(x, y)$ were in both $U_1$ and $W$ then since $(x, y) \in U_1$, $x = 0$, but since $(x, y) \in W$, $y = 0$ as well, so $(x, y) = (0, 0)$. Likewise, if $(x, y) \in U_2 \cap W$, then since $(x, y) \in W$, $y = 0$, but since $(x, y) \in U_2$, $x = y$, which means $x = 0$ as well. Thus $(x, y) = (0, 0)$ so $U_2 \cap W = \{(0, 0)\}$.

So we have shown that we can write $\mathbb{R}^2 = U_1 \oplus W$ and $\mathbb{R}^2 = U_2 \oplus W$. Since $U_1 \neq U_2$, this is our counterexample. □

**Question 1.** Let $X$ be a set, and let $W$ be a vector space over $\mathbb{F}$. Recall that $W^X$ is the set of functions $f : X \to W$. Given functions $f, g \in W^X$, and $a \in \mathbb{F}$, we define $f + g \in W^x$ and $a \cdot f \in W^X$ by

$$(f + g)(x) = f(x) + g(x)$$

$$(a \cdot f)(x) = a \cdot (f(x))$$

Prove that $W^X$ is a vector space over $\mathbb{F}$.

*Proof.* We have to show the properties of a vector space hold for vectors $f, g$, and $h$ in $W^X$ and scalars $a, b \in \mathbb{F}$. We will rely on the fact that $W$ itself is a vector space over $\mathbb{F}$, and so already satisfies all of these properties.

**Commutativity of Vector Addition:** We need to show that $f+g = g+f$. This is true since for any $x \in X$,

$$(f + g)(x) = f(x) + g(x)$$

and

$$(g + f)(x) = g(x) + f(x)$$

Since $W$ has commutativity of vector addition, $f(x) + g(x) = g(x) + f(x)$. Therefore $(f + g)(x) = (g + f)(x)$ for all $x \in X$. Since a function is determined by its values, this means that $f + g = g + f$.

**Associativity of Vector Addition:** We need to show that $f + (g + h) = (f + g) + h$. Again, we see what this means for the values of these functions. For any $x \in X$,

$$\begin{aligned}[f + (g + h)](x) &= f(x) + (g + h)(x) \\ &= f(x) + (g(x) + h(x))\end{aligned}$$

and

$$\begin{aligned}[(f + g) + h](x) &= (f + g)(x) + h(x) \\ &= (f(x) + g(x)) + h(x)\end{aligned}$$

We have that $f(x) + (g(x) + h(x)) = (f(x) + g(x)) + h(x)$ because $W$ is associative. Thus $f + (g + h)$ and $f + (g + h)$ have the same value for any $x \in X$, and therefore $f + (g + h) = (f + g) + h$.

**Additive Identity:** We need to show that there is a function $\mathbb{O} : X \to W$ so that $\mathbb{O} + f = f$ for all $f \in W^X$. Since $W$ is a vector space, it has an element $0 \in W$ where $0 + w = w$ for all $w \in W$. Let $\mathbb{O} : X \to W$ be the function defined by $\mathbb{O}(x) = 0$ for all $x \in X$. Then,

$$\begin{aligned}(\mathbb{O} + f)(x) &= \mathbb{O}(x) + f(x) \\ &= 0 + f(x) \\ &= f(x)\end{aligned}$$

where the last equality is true because 0 is the additive identity in $W$. Thus $\mathbb{O} + f = f$.

**Additive Inverse:** We need to show that for all $f \in W^X$, there is a $-f$ s.t. $f + (-f) = \mathbb{O}$. Since $W$ is a vector space, for all $w \in W$ there is a $-w \in W$ s.t. $w + (-w) = 0$. Thus given $f(x) \in W$ there is an element $-f(x) \in W$. Define the function $-f$ by the formula

$$(-f)(x) = -(f(x)).$$

Then for any $x \in X$,

$$\begin{aligned}(f + (-f))(x) &= f(x) + (-f)(x) \\ &= f(x) + (-(f(x))) \\ &= 0\end{aligned}$$

since $-f(x)$ is the additive inverse of $f(x)$ in $W$. Thus the function $f + (-f) = \mathbb{O}$, so each $f \in W^X$ has an additive inverse.

**Associativity of Scalar Multiplication:** We need to show that $a(b(f)) = (ab)(f)$. We use the definition of $af$, which says that $(af)(x) = a \cdot f(x)$. For any $x \in X$,

$$\begin{aligned}[a(b(f))](x) &= a \cdot (b(f)(x)) \\ &= a \cdot (b \cdot f(x)) \\ &= (ab) \cdot f(x) \text{ because } W \text{ has associativity of scalar mult.} \\ &= [(ab)(f)](x)\end{aligned}$$

Thus $a(b(f)) = (ab)(f)$.

**Distributivity of Scalar Sums:** We need to show that $(a+b)(f) = af+bf$. For any $x \in X$,

$$\begin{aligned}[(a+b)(f)](x) &= (a+b)\cdot f(x) \\ &= a\cdot f(x) + b\cdot f(x) \text{ since } W \text{ has distr. of scalar sums} \\ &= (af)(x) + (bf)(x)\end{aligned}$$

Thus $(a+b)(f) = af+bf$.

**Distributivity of Vector Sums:** We need to show that $a(f+g) = af+ag$. For any $x \in X$,

$$\begin{aligned}[a(f+g)](x) &= a\cdot (f+g)(x) \\ &= a\cdot (f(x)+g(x)) \\ &= a\cdot f(x) + a\cdot g(x) \text{ since } W \text{ is distr. over vector sums} \\ &= (af)(x) + (ag)(x)\end{aligned}$$

Thus $a(f+g) = af+ag$.

**Scalar Multiplication Identity:** We need to show that if $1 \in F$ is the multiplicative identity, then $1f = f$. For any $x \in X$,

$$\begin{aligned}(1f)(x) &= 1\cdot f(x) \\ &= f(x) \text{ since } 1\cdot w = w \text{ for all } w \in W\end{aligned}$$

Thus $1f = f$.

We have now shown that $W^X$ satisfies all of the properties of a vector space over $\mathbb{F}$. □

**Question 2.** Let $U$ be the following subset of $\mathbb{F}^\infty$:

$$U = \{(v_1, v_2, \dots) \in \mathbb{F}^\infty \mid v_{i+2} = v_i \text{for all i}\}$$

Prove that $U$ is a subspace of $F^\infty$.

*Proof.* To show that a subset $U$ of a vector space $\mathbb{F}^\infty$ is a subspace, we must show that $\mathbb{O} \in U$, and that $U$ is closed under vector addition and under scalar multiplication.

**Zero:** The vector $\mathbb{O} = (0, 0, \dots) \in \mathbb{F}^\infty$ satisfies the condition that $v_{i+2} = v_i$ for each $i$, since $v_i = 0$ and $v_{i+2} = 0$ for each $i$. Thus $\mathbb{O} \in U$.

**Closed Under Vector Addition:** We need to show that if $v, u \in U$ then $v + u \in U$. Suppose $v = (v_1, v_2, \dots)$ and $u = (u_1, u_2, \dots)$. Then $v + u = (v_1+u_1, v_2+u_2, \dots)$. That is, the $i^{th}$ entry of $u+v$ is $u_i+v_i$. Since $u_{i+2} = u_i$ and $v_{i+2} = v_i$, we have that $u_i + v_i = u_{i+2} + v_{i+2}$. Thus $u + v \in U$.

**Closed Under Scalar Multiplication:** We need to show that if $a \in \mathbb{F}$ and $u \in U$ then $a \cdot u \in U$. Let $u = (u_1, u_2, \dots)$. Then $a \cdot u = (a \cdot u_1, a \cdot u_2, \dots)$. That is, the $i^{th}$ entry of $a \cdot u$ is $a \cdot u_i$. If $u_i = u_{i+2}$, then $a \cdot u_i = a \cdot u_{i+2}$. Thus $a \cdot u \in U$.

We have shown that $U$ satisfies all of the conditions to be a subspace, so $U$ is a subspace of $F^\infty$. □

**Question 3.** Let $U_1, \dots, U_m$ be subspaces of a vector space $V$. The set $U_1, \dots, U_m$ is defined to be

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_i \in U_i \text{ for each } i = 1, \dots, m\}$$

Prove that $U_1 + \cdots + U_m$ is a subspace of $V$.

*Proof.* Let $U = U_1 + \cdots + U_m$. Once again, we need to show that $U$ satisfies the properties of a subspace. To do this, we will use the fact that each $U_i$ is itself a subspace.

**Zero:** Since $U_i$ is a subspace for each $i$, we have that $0 \in U_i$ for each $i$. Thus $0 + 0 + \cdots + 0 = 0$, which shows that $0 \in U$.

**Closed Under Vector Addition:** We need to show that if $v, u \in U$ then $v + u \in U$. Since $v \in U$, we can write $v = v_1 + v_2 + \dots v_m$, where $v_i \in U_i$. Likewise, we can write $u = u_1 + u_2 + \cdots + u_m$ for $u_i \in U_i$. Then $v + u = (v_1 + u_1) + (v_2 + u_2) + \cdots + (v_m + u_m)$ since addition in the bigger vector space $V$ is associative. Since each $U_i$ is a subspace of $V$, and since $v_i$ and $u_i \in U$, we have that $v_i + u_i \in U$. Setting $w_i = v_i + u_i$, we have shown that $v + u = w_1 + w_2 + \cdots + w_m$ for $w_i \in U_i$. Thus $v + u \in U$.

**Closed Under Scalar Multiplication:** We need to show that if $a \in \mathbb{F}$ and $u \in U$ then $a \cdot u \in U$. Let $u = u_1 + u_2 + \cdots + u_m \in U$. Then $a \cdot u = a \cdot u_1 + a \cdot u_2 + \cdots + a \cdot u_m$. Since $U_i$ is a subspace of $V$, and since $u_i \in U_i$, we have that $a \cdot U_i \in U_i$. Thus we can write $a \cdot u$ as a sum of terms in the $U_i$. Therefore, $a \cdot u \in U$.

Since $U$ satisfies the conditions to be a subspace, it is a subspace of $V$. □

**Question 4.** Let $U_1 = \{(a, 0, 0) \mid a \in \mathbb{F}\}$ and $U_2 = \{(b, b, 0) \mid b \in \mathbb{F}\}$. These are both subsets of $\mathbb{F}^3$.

(1) Prove that $U_1$ and $U_2$ are subspaces of $\mathbb{F}^3$.
(2) Prove that $U_1 + U_2 = \{(x, y, 0) \mid x, y \in \mathbb{F}\}$.

*Proof.* (1) First we show that $U_1$ and $U_2$ are subspace of $\mathbb{F}^3$. We do the usual steps.

**Zero:** Note that $(0, 0, 0)$ is the zero in $\mathbb{F}^3$, and it is in both $U_1$ (with $a = 0$) and $U_2$ (with $b = 0$).

**Closed Under Vector Addition:** We need to show that if $v, u \in U_i$ then $v + u \in U_i$ for $i = 1, 2$. Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be elements of $\mathbb{F}^3$. Then $a + b = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$. If $a_3 = b_3 = 0$, then the last term of $a + b$ is zero. The same holds for the case when $a_2 = b_3 = 0$. Thus if $a$ and $b$ were in $U_1$, then $a + b$ would be in $U_1$. If $a_1 = a_2$ and $b_1 = b_2$ then $a_1 + b_1 = a_2 + b_2$. Thus if $a, b \in U_2$ then $a + b \in U_2$. Therefore, both $U_1$ and $U_2$ are closed under addition.

**Closed Under Scalar Multiplication:** We need to show that if $x \in \mathbb{F}$ and $u \in U_i$ for $i = 1, 2$ then $x \cdot u \in U_i$. Note that if $a = (a_1, a_2, a_3)$ then $x \cdot a = (xa_1, xa_2, xa_3)$. If $a_3 = 0$ then $xa_3 = 0$, and the same is true for $a_2$. Thus, if $a \in U_1$, then $x \cdot a \in U_1$. If, on the other hand, $a_1 = a_2$ then $xa_1 = xa_2$. Thus if $a \in U_2$ then $x \cdot a \in U_2$.

Therefore, $U_1$ and $U_2$ are subspaces of $\mathbb{F}^3$.

(2) Now we show that $U_1 + U_2 = \{(x, y, 0) \mid x, y \in \mathbb{F}\}$. Let $x, y \in \mathbb{F}$. We will show that $(x, y, 0) \in U_1 + U_2$ by finding elements of $U_1$ and $U_2$ that add up

to $(x, y, 0)$. Let $a = x + (-y) \in \mathbb{F}$ and let $b = y$. Then $(a, 0, 0) \in U_1$ and $(b, b, 0) \in U_2$. Thus $a + b = (a + b, b, 0)$ where

$$\begin{aligned}(a + b, b, 0) &= (x + (-y) + y, y, 0) \\ &= (x, y, 0)\end{aligned}$$

using the rules of field addition. Thus $(x, y, 0) \in U_1 + U_2$. Therefore, $U_1 + U_2 = \{(x, y, 0) \mid x, y \in \mathbb{F}\}$.

□

**Question 5.** Let $U_1 = \{(a, -a, 0) \mid a \in \mathbb{F}\}$, let $U_2 = \{(0, b, -b) \mid b \in \mathbb{F}\}$ and let $U_3 = \{(c, 0, -c) \mid c \in \mathbb{F}\}$. These are all subspaces of $\mathbb{F}^3$.

(1) Describe the subspace $U_1 + U_2 + U_3$.
(2) Let $W = U_1 + U_2 + U_3$. Is $W$ the direct sum of $U_1, U_2$ and $U_3$? Prove or disprove.

*Proof.* (1) First we describe the subspace $W = U_1 + U_2 + U_3$. We will show that

$$W = \{(x, y, z) \in \mathbb{F}^3 | x + y + z = 0\}.$$

Let $A$ be any vector in $U_1$, let $B$ be any vector in $U_2$, and let $C$ be any vector in $U_3$. We can write $A = (a, -a, 0) \in U_1$, $B = (0, b, -b) \in U_2$ and $C = (c, 0, -c) \in U_3$ for some $a, b, c \in \mathbb{F}$. Then any vector in $U_1 + U_2 + U_3$ is of the form

$$A + B + C = (a + c, b - a, -c - b)$$

Note that $(a+c)+(b-a)+(-c-b) = (a-a)+(b-b)+(c-c) = 0+0+0 = 0$, so every such vector is contained in $\{(x, y, z)|x + y + z = 0\}$. This shows that $U_1+U_2+U_3 \subset \{(x, y, z)|x+y+z = 0\}$. Conversely, if $(x, y, z)$ satisfies $x+y+z = 0$, we can rearrange this as $y = -x-z$. Therefore we can write

$$(x, y, z) = (x, -x - z, z) = (x, -x, 0) + (0, -z, z) + (0, 0, 0).$$

Since $(x, -x, 0) \in U_1$, $(0, -z, z) \in U_2$, and $(0, 0, 0) \in U_3$, this shows that $\{(x, y, z)|x + y + z = 0\} \subset U_1 + U_2 + U_3$.

(2) We will disprove that $W$ is the direct sum of $U_1, U_2$ and $U_3$. To disprove this, we need to find an element $(x, y, z)$ of $W$ that can be written in two ways as a sum of element of $U_1, U_2$ and $U_3$.

We will use $(x, y, z) = (0, 0, 0)$. Then we can write

$$(0, 0, 0) = (1, -1, 0) + (0, 1, -1) + (-1, 0, 1)$$

where $(1, -1, 0) \in U_1$, $(0, 1, -1) \in U_2$ and $(-1, 0, 1) \in U_3$. However, we can also write

$$(0, 0, 0) = (0, 0, 0) + (0, 0, 0) + (0, 0, 0).$$

Since certainly $(0, 0, 0) \in U_1$, $(0, 0, 0) \in U_2$, and $(0, 0, 0) \in U_3$, this shows that $(0, 0, 0)$ can be written in two different ways as the sum of elements from $U_1$, $U_2$, and $U_3$. Thus $W$ is not the direct sum of $U_1$, $U_2$, and $U_3$. [2]

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[2]Editor's note (TC): If your field is $\mathbb{R}$ or $\mathbb{C}$, there are tons of other examples, e.g. you could let $(x, y, z) = (2, 0, -2)$. Then we could write (footnote continued on next page)

$$(2, 0, -2) = (1, -1, 0) + (0, 1, -1) + (1, 0, -1)$$

where $(1, -1, 0) \in U_1$, $(0, 1, -1) \in U_2$ and $(1, 0, -1) \in U_3$ but also that

$$(2, 0, -2) = (3, -3, 0) + (0, 3, -3) + (-1, 0, 1)$$

□

**Question 6.** Let $V$ be a vector space, and let $\mathcal{U}$ be a collection of subspaces of $V$. The set $\bigcap \mathcal{U}$ is

$$\bigcap \mathcal{U} = \{v \in V \mid v \in U \text{ for every } U \in \mathcal{U}\}$$

(1) Prove that $\bigcap \mathcal{U}$ is a subspace of $V$.
(2) Now let $W_1, \dots, W_m$ be subspaces of $V$, and let $\mathcal{U}$ be the collection of subspaces containing each $W_i$. That is,

$$\mathcal{U} = \{U \subset V \mid U \text{is a subspace of V, and } W_i \subset U \text{ for each } i = 1, \dots, m\}$$

Prove that in this case $\bigcap \mathcal{U} = W_1 + \cdots + W_m$.

*Proof.* (1) First we prove that $\bigcap \mathcal{U}$ is a subspace of $V$.

Once again, we show the following three properties.

**Zero:** Since each element of $\mathcal{U}$ is a subspace of $V$, we have that $0 \in U$ for each $U \in \mathcal{U}$. Thus $0 \in \bigcap \mathcal{U}$.

**Closed Under Vector Addition:** We need to show that if $v, u \in \bigcap \mathcal{U}$ then $v + u \in \bigcap \mathcal{U}$. If $v, u \in \bigcap \mathcal{U}$ then $v, u \in U$ for each $U \in \mathcal{U}$. Each $U \in \mathcal{U}$ is a subspace of $V$. Thus if $u, v \in U$ then $u + v \in U$. So $u + v \in U$ for each $U \in \mathcal{U}$. Therefore $u + v \in \bigcap \mathcal{U}$.

**Closed Under Scalar Multiplication:** We need to show that if $x \in \mathbb{F}$ and $u \in \bigcap \mathcal{U}$ then $x \cdot u \in \bigcap \mathcal{U}$. Again, if $u \in \bigcap \mathcal{U}$, then $u \in U$ for each vector space $U \in \mathcal{U}$. Thus $x \cdot u \in U$ for each $U \in \mathcal{U}$. Therefore, $x \cdot u \in \bigcap \mathcal{U}$.

Therefore, $\bigcap \mathcal{U}$ is a subspace of $V$.

(2) Let $W_1, \dots, W_m$ be subspaces of $V$, and let $\mathcal{U}$ be the collection of subspaces containing each $W_i$. That is,

$$\mathcal{U} = \{U \subset V \mid U \text{is a subspace of V, and } W_i \subset U \text{ for each } i = 1, \dots, m\}$$

First we want to show that $\bigcap \mathcal{U} \subset W_1 + \cdots + W_m$.

We will first show that $W_1 + \cdots + W_m$ contains $W_i$ for each $i$. Suppose $w \in W_i$. Since $W_1, \dots, W_m$ are all subspaces, they all contain 0. So we can write $w$ as an element of $W_1 + \cdots + W_m$ by writing $w = 0 + \cdots + 0 + w + 0 + \cdots + 0$ where $w$ is in the $i^{th}$ position. Thus for each $w \in W_i$, $w$ is also in $W_1 + \cdots + W_m$. Therefore, $W_1 + \cdots + W_m$ contains $W_i$ for each $i$.

Now suppose that $u \in \bigcap \mathcal{U}$. Then $u$ is contained in any subspace $U$ that contains each $W_i$. Since $W_1 + \cdots + W_m$ contains $W_i$ for each $i$, $u$ is, in particular, contained in $W_1 + \cdots + W_m$. Thus each $u \in \bigcap \mathcal{U}$ is contained in $W_1 + \cdots + W_m$. Therefore, $\bigcap \mathcal{U} \subset W_1 + \cdots + W_m$.

Next we show that $W_1 + \cdots + W_m \subset \bigcap \mathcal{U}$. Suppose $w_1 + \cdots + w_m \in W_1 + \cdots + W_m$ where $w_i \in W_i$ for each $i$. Take a subspace $U$ that contains $W_i$

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where $(3, -3, 0) \in U_1$, $(0, 3, -3) \in U_2$ and $(-1, 0, 1) \in U_3$.

However, although this works for $\mathbb{R}$ and $\mathbb{C}$, this would not be a completely valid proof for a general field $\mathbb{F}$, because the integers 2 and 3 might not be elements of our field $\mathbb{F}$. You might well think "Well, I know that there's an element of $\mathbb{F}$ called 1, so I could define $2 :- 1 + 1$ and $3 :- 1+1+1$." And indeed you can — but this might not behave the way you expect. For example, in the field $\mathbb{F}_2 = \{0, 1\}$, we saw that $1 + 1 = 0$ — so I guess $2 = 0$? and $3 = 1$? Moreover, the equation $1 + 1 = 0$ shows that 1 is its *own* additive inverse in $\mathbb{F}_2$, so we also have $-1 = 1$! As a result, both of the equations above for $(2, 0, -2)$ would actually be the *same* if our field was $\mathbb{F}_2$.

Although mathematically this is a very important point, you don't need to worry if you made this mistake on the homework. I am happy to talk about it if you're interested, though. -TC for each $i$. Then $w_i$ is in $U$ for each $i$. Since $U$ is a subspace, $w_1 + \cdots + w_m \in U$. Thus each subspace that contains $W_i$ for each $i$ must also contain $w_1 + \cdots + w_m$. Therefore, by definition of $\bigcap \mathcal{U}$, each element of $W_1 + \cdots + W_m$ is an element of $\bigcap \mathcal{U}$. Therefore $W_1 + \cdots + W_m \subset \bigcap \mathcal{U}$.

Since $\bigcap \mathcal{U} \subset W_1 + \cdots + W_m$ and $W_1 + \cdots + W_m \subset \bigcap \mathcal{U}$, we have that $W_1 + \cdots + W_m = \bigcap \mathcal{U}$.

□