Math 113 – Winter 2013 – Prof. Church
Final Exam: due Monday, March 18 at 3:15pm

Name: ______________________________________

Student ID: ________________________________

Signature: _________________________________

Your exam should be turned in to me in my office, 383-Y (third floor of the math building). If I
am not there, slide your exam under the door. Your exam **must** be handed in by 3:15pm or you will
receive a zero.

This exam is open-book and open-notes, but closed-everything-else. (Needless to say, you should
not discuss this exam with anyone.) In your proofs you may use any theorem from class; from Chapters
1–7 of Axler (plus Theorems 8.34 and 8.36); or from the notes on wedge vectors and determinants
(available on my website). You may read the homework/midterm solutions if you like (also on my
website), but you cannot quote them as a reference. When giving counterexamples, you may describe
your operators either by a formula or by a matrix.

There are 5 questions worth 100 points total on this exam, plus a 10-point bonus question; you
should finish the other questions before attempting the bonus question.

Questions? E-mail Prof. Church at church@math.stanford.edu.

<table>
<thead>
<tr>
<th>1a</th>
<th>1b</th>
<th>2a</th>
<th>2b</th>
<th>3a</th>
<th>3b</th>
<th>4a</th>
<th>4b</th>
<th>4c</th>
<th>5a</th>
<th>5b</th>
<th>5c</th>
<th>5d</th>
<th>Bonus</th>
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<tbody>
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**Question 1** (20 points). Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, and let
$S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(V)$ be operators on $V$ satisfying $ST = 0$.

1a) Prove that $\text{Image} T \subset \text{Null} S$. 
Recall our assumptions: $V$ is finite-dimensional;
\[ S \in \mathcal{L}(V), \ T \in \mathcal{L}(V); \]
\[ ST = 0. \]

1b) For each of the following assertions, either prove that it must hold, or give a counterexample.

I. Either $S = 0$ or $T = 0$.

II. $TS = 0$.

III. If $\det(S) = 6$, then $T = 0$.

IV. There exists a nonzero $v \in V$ such that $TS(v) = 0$. 
**Question 2** (15 points). Let $V = \mathbb{R}^2$, and let $T \in \mathcal{L}(V)$ be an operator on $V$. Assume that $v \in V$ and $w \in V$ are two nonzero vectors satisfying

$$T(v) = 2v \quad \text{and} \quad T(w) = -w.$$ 

2a) Compute the determinant $\det(T^4 + T)$.

2b) Do we have enough information to determine the minimal polynomial $m_T(x)$? If so, find the minimal polynomial; if not, explain why not.
**Question 3** (20 points). Let $V$ and $W$ be finite-dimensional vector spaces over $\mathbb{R}$. Assume that $Q \in \mathcal{L}(V, W)$, $R \in \mathcal{L}(V, W)$, and $S \in \mathcal{L}(V, W)$ are each rank-1 transformations:

$$\text{rank}(Q) = 1 \quad \text{rank}(R) = 1 \quad \text{rank}(S) = 1$$

We’ll be considering the transformation $Q + R + S \in \mathcal{L}(V, W)$, so let’s give it a name: let $T \in \mathcal{L}(V, W)$ be the transformation

$$T = Q + R + S \in \mathcal{L}(V, W)$$

3a) Prove that $\text{rank}(Q + R + S) \leq 3$. 
Recall our assumptions: $V$ and $W$ are finite-dimensional;
\[ Q, R, S \in \mathcal{L}(V, W) \]
\[ \text{rank}(Q) = 1, \text{rank}(R) = 1, \text{rank}(S) = 1; \]
\[ T = Q + R + S. \]

3b) Is the following assertion (*) true?

\[ \text{rank}(Q + R + S) < 3 \iff (Q, R, S) \text{ are linearly dependent in } \mathcal{L}(V, W) \quad (*) \]

Prove or give a counterexample.
Question 4 (20 points). Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, and let $S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(V)$ be operators on $V$ satisfying $S^2 = S$ and $T^2 = T$. Additionally, assume that $S + T = I$.

4a) Prove that $ST = 0$. 
Recall our assumptions: \( V \) is finite-dimensional; \( S, T \in \mathcal{L}(V) \); \( S^2 = S \) and \( T^2 = T \); \( S + T = I \).

Let \( U = \text{Image } S \) and \( W = \text{Image } T \).

4b) Prove that \( W = \text{Null } S \) and \( U = \text{Null } T \).
Recall our assumptions: $V$ is finite-dimensional; $S, T \in \mathcal{L}(V);$ $S^2 = S$ and $T^2 = T;$ $S + T = I;$ $U = \text{Image } S$ and $W = \text{Image } T.$

4c) Prove that $V = U \oplus W.$
**Question 5** (25 points). If $V$ is an inner product space over $\mathbb{R}$ or $\mathbb{C}$, we define an operator $S \in \mathcal{L}(V)$ to be *skew-self-adjoint* if it is equal to the *negative* of its adjoint:

$$S^* = -S$$

5a) Prove that *every* operator $R \in \mathcal{L}(V)$ can be written as a sum $R = T + S$ where $T \in \mathcal{L}(V)$ is self-adjoint and $S \in \mathcal{L}(V)$ is skew-self-adjoint.
For the remaining parts, assume that $V$ is an inner product space over $\mathbb{R}$, and $S \in \mathcal{L}(V)$ is skew-self-adjoint.

5b) Prove that if $S$ is injective, then $S$ has no eigenvectors.
Recall our assumptions: \( V \) is a finite-dimensional inner product space over \( \mathbb{R} \),
\( S \in \mathcal{L}(V) \) is skew-self-adjoint.

5c) Prove that the operator \( S^2 \in \mathcal{L}(V) \) is diagonalizable.
5d) Let $SSA(V) \subset \mathcal{L}(V)$ be the subspace of skew-self-adjoint operators (you do not need to prove that this is a subspace).

Let $V$ be a 3-dimensional inner product space over $\mathbb{R}$ with orthonormal basis $v_1, v_2, v_3$. Find an explicit basis for $SSA(V)$. What is the dimension of $SSA(V)$?
**Question 6** (Bonus question, 10 points). Let $V$ be a finite-dimensional inner product space over $\mathbb{C}$, and let $S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(V)$ be operators on $V$.

6a) Assume that $S$ and $T$ are self-adjoint operators. Prove that if $ST = TS$, then there exists an orthonormal basis $v_1, \ldots, v_n$ of $V$ so that each basis vector $v_i$ is both an eigenvector of $S$ and an eigenvector of $T$. 
Recall our assumption: $V$ is a finite-dimensional inner product space over $\mathbb{C}$.

6b) If we only assume that $S$ and $T$ are normal operators satisfying $ST = TS$, is it true that there exists an orthonormal basis $v_1, \ldots, v_n$ of $V$ so that each basis vector $v_i$ is both an eigenvector of $S$ and an eigenvector of $T$?

Either prove this or give a counterexample.