Math 113 Homework 6 Solutions

Solutions by Guanyang Wang, with edits by Tom Church. Exercises from the book.

Exercise 6.C.4 Suppose $U$ is the subspace of $\mathbb{R}^4$ defined by

$$U = \text{span}((1, 2, 3, -4), (-5, 4, 3, 2))$$

Find an orthonormal basis of $U$ and an orthonormal basis of $U^\perp$.

Answer. Notice that the list $(1, 2, 3, -4)$ and $(-5, 4, 3, 2)$ is linearly independent since neither vector is the scalar multiple of the other. Thus we will extend the list $(1, 2, 3, -4), (-5, 4, 3, 2)$ to a basis

$$(1, 2, 3, -4), (-5, 4, 3, 2), w_1, w_2$$

of $\mathbb{R}^4$ and then apply the Gram-Schmidt Procedure.

To extend $(1, 2, 3, -4), (-5, 4, 3, 2)$ to a basis of $\mathbb{R}^4$, we follow the idea of the proof of 2.33. Thus we start with the list

$$(1, 2, 3, -4), (-5, 4, 3, 2), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$$

which spans $\mathbb{R}^4$. We need to apply the Gram-Schmidt Procedure anyway, and thus in this case the easiest thing to do is to start the Gram-Schmidt Procedure and throw out any vectors that would lead to division by 0 (indicating linear independence), or stop when we reach a list of length four.

To get started, we have

$$e_1 = \frac{(1, 2, 3, -4)}{\| (1, 2, 3, -4) \|} = \left( \frac{1}{\sqrt{30}}, \sqrt{\frac{2}{15}}, \sqrt{\frac{3}{10}}, -\sqrt{\frac{2}{15}} \right).$$

Next,

$$e_2 = \frac{(-5, 4, 3, 2) - \langle (-5, 4, 3, 2), e_1 \rangle e_1}{\| (-5, 4, 3, 2) - \langle (-5, 4, 3, 2), e_1 \rangle e_1 \|}$$

$$= \left( -\frac{77}{\sqrt{12030}}, \frac{28}{6015}, 13\sqrt{\frac{2}{15}}, 19\sqrt{\frac{2}{6015}} \right).$$

Next,

$$e_3 = \frac{(1, 0, 0, 0) - \langle (1, 0, 0, 0), e_1 \rangle e_1 - \langle (1, 0, 0, 0), e_2 \rangle e_2}{\| (1, 0, 0, 0) - \langle (1, 0, 0, 0), e_1 \rangle e_1 - \langle (1, 0, 0, 0), e_2 \rangle e_2 \|}$$

$$= \left( \frac{190}{401}, \frac{117}{76190}, 6\sqrt{\frac{10}{7619}}, \frac{151}{\sqrt{76190}} \right).$$

There is no division by 0 here, and no linear dependence yet.

Next,

$$e_4 = \frac{(0, 1, 0, 0) - \langle (0, 1, 0, 0), e_1 \rangle e_1 - \langle (0, 1, 0, 0), e_2 \rangle e_2 - \langle (0, 1, 0, 0), e_3 \rangle e_3}{\| (0, 1, 0, 0) - \langle (0, 1, 0, 0), e_1 \rangle e_1 - \langle (0, 1, 0, 0), e_2 \rangle e_2 - \langle (0, 1, 0, 0), e_3 \rangle e_3 \|}$$

$$= \left( 0, \frac{9}{\sqrt{190}}, -\sqrt{\frac{10}{19}}, -\frac{3}{\sqrt{190}} \right).$$

Again there is no division by 0 here, and thus no linear dependence yet.

Since $\mathbb{R}^4$ has dimension 4, we know that $e_1, e_2, e_3, e_4$ is a basis of $\mathbb{R}^4$, and there is no need to continue the process further.
Thus, by the previous exercise,
\[
\left(\frac{1}{\sqrt{30}}, \sqrt{\frac{2}{15}}, \sqrt{\frac{3}{10}}, -2\sqrt{\frac{2}{15}}\right), \left(-\frac{77}{\sqrt{12030}}, \frac{28}{6015}, \frac{13}{\sqrt{4010}}, \frac{19}{\sqrt{6015}}\right)
\]

is an orthonormal basis of \(U\) and
\[
\left(\frac{190}{401}, \frac{117}{\sqrt{76190}}, \frac{6}{\sqrt{76190}}, -\frac{151}{\sqrt{76190}}\right), \left(0, \frac{9}{\sqrt{190}}, -\frac{\sqrt{10}}{19}, -\frac{3}{\sqrt{190}}\right)
\]
is an orthonormal basis of \(U^\perp\).

\[\square\]

**Exercise 6.C.6** Suppose \(U\) and \(W\) are finite-dimensional subspaces of \(V\). Prove that \(P_UP_W = 0\) if and only if \(\langle u, w \rangle = 0\) for all \(u \in U\) and all \(w \in W\).

*Proof.* First suppose \(P_UP_W = 0\). Suppose \(w \in W\). Then
\[
0 = P_UP_W w = P_U w
\]
Hence \(w \in \text{null}P_U\). Now 6.55(e) shows that \(w \in U^\perp\). Thus \(\langle u, w \rangle = 0\) for all \(u \in U\), completing one direction of the proof.

To prove the other direction, now suppose that \(\langle u, w \rangle = 0\) for all \(u \in U\) and all \(w \in W\). Thus \(U \subset W^\perp\) and \(W \subset U^\perp\). If \(w \in W\), then
\[
(P_UP_W)(w) = P_U(P_W w) = P_U w = 0
\]
where the last equality holds because \(w \in U^\perp\). If \(v \in W^\perp\), then
\[
(P_UP_W)(v) = P_U(P_W v) = P_U v = 0.
\]

Since every element in \(V\) can be written as the sum of a vector in \(W\) and a vector in \(W^\perp\) (by 6.47), the last two equations imply that \(P_UP_W = 0\), as desired. \(\square\)

**Exercise 6.C.11** In \(\mathbb{R}^4\), let
\[U = \text{span}((1, 1, 0, 0), (1, 1, 1, 2))\]
Find \(u \in U\) s.t. \(||u - (1, 2, 3, 4)||\) is as small as possible.

*Proof.* First, we find an orthogonal basis for \(U\). (So we won’t bother to make the vectors have norm 1.) We keep \(u_1 = (1, 1, 0, 0)\). Then we subtract off \(u_1\) from \(v_2 = (1, 1, 1, 2)\). We have that \(v_2 - u_1 = (0, 0, 1, 2)\) is perpendicular to \(u_1\). So we set \(u_2 = (0, 0, 1, 2)\). Now \(u_1, u_2\) form a basis for \(U\). Using this basis, we see that elements of \(U\) are vectors of the form \((x, x, y, 2y)\) for \(x, y \in \mathbb{R}\).

So we want to find \(x\) and \(y\) s.t. the vector \((x, x, y, 2y) - (1, 2, 3, 4)\) has the least norm. Noting that \((x, x, y, 2y) - (1, 2, 3, 4) = (x-1, x-2, y-3, 2y-4)\), we compute
\[
||(x-1, x-2, y-3, 2y-4)||^2 = (x-1)^2 + (x-2)^2 + (y-3)^2 + (2y-4)^2
= 2x^2 - 6x + 5y^2 - 22y + 16
= 2x^2 - 6x + 5y^2 - 22y + 21
\]
This is minimized when \(p(x) = 2x^2 - 6x\) and \(q(y) = 5y^2 - 22y\) are both minimized. As their leading coefficients are positive, both of these quadratics go to infinity as \(x\).
and $y$ go to infinity, respectively. Thus their local critical points are their respective minima. Taking derivatives, we get that

$$p'(x) = 4x - 6 \quad \text{and} \quad q'(y) = 10y - 22$$

So their minima are at $x = \frac{3}{2}$ and $y = \frac{11}{5}$, respectively. Therefore the vector $u \in U$ s.t. $\|u - (1, 2, 3, 4)\|$ is smallest is $u = (\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5})$.

Here is another way to do this problem:

The vector $v_3 = (0, 1, 1, 0)$ is not in $U$ because it is not of the correct form. Note that $(1, 2, 3, 4) = u_1 + 2u_2 + v_3$ so $(1, 2, 3, 4)$ is in the span of $u_1, u_2$ and $v_3$. We want to find a vector $u_3$ in the span of $u_1, u_2$ and $v_3$ s.t. $u_3$ is orthogonal to $u_1$ and $u_2$. We have that

$$v_3 \cdot u_1 = 1 \quad \text{and} \quad v_3 \cdot u_2 = 1$$

We also have

$$u_1 \cdot u_1 = 2 \quad \text{and} \quad u_2 \cdot u_2 = 5$$

Thus $u_3 = v_3 - \frac{1}{2}u_1 - \frac{1}{5}u_2$ is orthogonal to $u_1$ and $u_2$. We can see this directly by writing $u_3 = (-\frac{1}{2}, 1, 4, -\frac{2}{5})$. Since $(1, 2, 3, 4) = u_1 + 2u_2 + v_3$ and $v_3 = u_3 + \frac{1}{2}u_1 + \frac{1}{5}u_2$, we get that

$$(1, 2, 3, 4) = \frac{3}{2}u_1 + \frac{11}{5}u_2 + u_3$$

$$= \frac{3}{2}(1, 1, 0, 0) + \frac{11}{5}(0, 0, 1, 2) + (-\frac{1}{2}, \frac{1}{2}, \frac{4}{5}, -\frac{2}{5})$$

Now suppose $u \in U$ is the vector s.t. $\|u - (1, 2, 3, 4)\|$ is minimal. Since $u \in U$, we can write $u = a_1u_1 + a_2u_2$ for some $a_1, a_2 \in \mathbb{R}$. Thus,

$$\|u - (1, 2, 3, 4)\| = \|a_1u_1 + a_2u_2 - \frac{3}{2}u_1 + \frac{11}{5}u_2 + u_3\|$$

$$= \|(a_1 - \frac{3}{2})u_1 + (a_2 - \frac{11}{5})u_2 - u_3\|$$

$$= (a_1 - \frac{3}{2})^2\|u_1\|^2 + (a_2 - \frac{11}{5})^2\|u_2\|^2 + \|u_3\|^2$$

because $u_1, u_2, u_3$ are orthogonal. This quantity is minimized when $a_1 = \frac{3}{2}$ and $a_2 = \frac{11}{5}$. Thus the $u \in U$ that is closest to $(1, 2, 3, 4)$ is $\frac{3}{2}u_1 + \frac{11}{5}u_2 = (\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5})$. □

**Exercise 7.A.1.** Suppose $n$ is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(z_1, ..., z_n) = (0, z_1, ..., z_{n-1})$$

Find a formula for $T^*(z_1, ..., z_n)$

**Proof.** Fix $(z_1, ..., z_n) \in \mathbb{F}^n$. Then for every $(w_1, ..., w_n) \in \mathbb{F}^n$, we have

$$\langle (w_1, ..., w_n), T^*(z_1, ..., z_n) \rangle = \langle T(w_1, ..., w_n), (z_1, ..., z_n) \rangle$$

$$= \langle (0, w_1, ..., w_{n-1}), (z_1, ..., z_n) \rangle$$

$$= w_1z_2 + ... + w_{n-1}z_n$$

$$= \langle (w_1, ..., w_n), (z_2, ..., z_n, 0) \rangle.$$
Exercise 7.A.2 Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that $\lambda$ is an eigenvalue of $T$ iff $\overline{\lambda}$ is an eigenvalue of $T^*$.

Proof. Suppose $\lambda$ is an eigenvalue of $T$. Then there is some $v \neq 0$ s.t. $Tv = \lambda v$. Thus,

$$\langle Tv, w \rangle = \langle \lambda v, w \rangle$$

for each $w \in V$. Note that $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle = \langle v, \overline{\lambda}w \rangle$ by linearity of the inner product. If $T^*$ is the adjoint of $T$, then $\langle T(v), w \rangle = \langle v, T^*w \rangle$, so we have

$$\langle v, \overline{\lambda}w \rangle = \langle v, T^*w \rangle$$

for each $w$ in $W$. By linearity of the inner product, this means that

$$\langle v, T^*w - \overline{\lambda}w \rangle = 0$$

for each $w \in W$. Thus, $v$ is perpendicular to all vectors of the form $T^*w - \overline{\lambda}w$.

Let $S = T^* - \overline{\lambda}I$. The image of $S$ is all vectors of the form $T^*w - \overline{\lambda}w$ so $v$ is perpendicular to all vectors in the image of $S$. However, since $v$ is nonzero, it cannot be perpendicular to itself ($\langle v, v \rangle > 0$ is an axiom of inner products), so $v \notin \text{Image} S$. This shows that $\text{Image} S \neq V$, so $\dim \text{Image} S$ must be strictly less than the dimension of the image of $V$. By Rank-Nullity, this implies that $\dim \text{Null} S > 0$. Therefore, there is some non-zero element in the null space of $S$. So there is some $w$ s.t. $T^*w = \overline{\lambda}w$, meaning $\overline{\lambda}$ is an eigenvalue of $T^*$.

Since $(T^*)^* = T$, this also shows that any eigenvalue of $T^*$ is also the conjugate of an eigenvalue of $T$. Therefore $\lambda$ is an eigenvalue of $T$ iff $\overline{\lambda}$ is an eigenvalue of $T^*$. \hfill \Box

Exercise 7.A.4 Suppose $T \in \mathcal{L}(V,W)$. Prove that

(a) $T$ is injective if and only if $T^*$ is surjective;
(b) $T$ is surjective if and only if $T^*$ is injective.

Proof. First we prove (a)

$T$ is injective $\iff$ Null $T = 0$
\[\iff (\text{Range} T^*)^\perp = 0\]
\[\iff \text{Range} T^* = W\]
\[\iff T$ is surjective

Where the second line comes from 7.7(c).

Note that (a) has been proved, (b) follows immediately by replacing $T$ with $T^*$ in (a). \hfill \Box

Question 1. Suppose $(e_1, \ldots, e_m)$ is an orthonormal list of vectors in $V$. Let $v \in V$. Prove that

$$||v||^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$$

if and only if $v \in \text{span}(e_1, \ldots, e_m)$. 

Proof. Denote \( \text{span}(e_1, \ldots, e_m) \) by \( U \). Then we can write every vector \( v \) in \( V \) as \( u + w \) with \( u \in U \) and \( w \in U^\perp \). So we have

\[
||v||^2 = \langle u + w, u + w \rangle = \langle u, u \rangle + \langle w, w \rangle = ||u||^2 + ||w||^2
\]

The second equality holds since \( \langle u, w \rangle = \langle w, u \rangle = 0 \).

Then, since \((e_1, \ldots, e_m)\) is an orthonormal basis of \( U \), we can find \( a_1, \ldots, a_m \) such that \( u = a_1e_1 + \cdots + a_m e_m \). Therefore we have

\[
||u||^2 = \langle a_1e_1 + \cdots + a_m e_m, a_1e_1 + \cdots + a_m e_m \rangle
\]

\[
= \sum_{i,j=1}^{m} \langle e_i, e_j \rangle a_i a_j
\]

Since \( e_i \) and \( e_j \) are orthogonal if \( i \neq j \), and since \( \langle e_i, e_i \rangle = 1 \), we get

\[
||u||^2 = |a_1|^2 + \cdots + |a_m|^2
\]

On the other hand,

\[
\langle v, e_i \rangle = \langle u + w, e_i \rangle = \langle a_1 e_1 + \cdots + a_m e_m + w, e_i \rangle = a_i
\]

Since \( e_i \) and \( w \) are orthogonal for every \( i \in \{1, 2, \ldots, m\} \), and since \( e_i \) and \( e_j \) are orthogonal if \( i \neq j \).

So, \( |\langle v, e_i \rangle|^2 = |a_i|^2 \) meaning that

\[
||v||^2 = ||u||^2 + ||w||^2
\]

\[
= |a_1|^2 + \cdots + |a_m|^2 + ||w||^2
\]

\[
= |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 + ||w||^2
\]

If \( v \in \text{span}(e_1, \ldots, e_m) \), then \( v = a_1 e_1 + \cdots + a_m e_m \). That is, \( w = 0 \). Thus \( ||v||^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 \).

If \( ||v||^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 \), then we have \( ||w||^2 = 0 \), therefore \( w = 0 \), so we have \( v = u + 0 = u \in U \). By definition, \( U = \text{span}(e_1, \ldots, e_m) \), so \( v \in \text{span}(e_1, \ldots, e_m) \).

\( \square \)

**Question 2.** Let \( V \) be the vector space of infinite sequences of real numbers:

\[
V = \{(a_1, a_2, \ldots) \mid a_i \in \mathbb{R} \}
\]

This is an infinite dimensional vector space over \( \mathbb{R} \). Let \( T \in \mathcal{L}(V) \) be the forward shift defined by

\[
T(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)
\]
a) The operator $T + I$ is given by

$$(T + I)(a_1, a_2, a_3, \ldots) = (a_1 + 1, a_2 + 1, a_3 + 1, \ldots)$$

Find an inverse $(T + I)^{-1}$ for this operator.
b) For which values $\lambda \in \mathbb{R}$ is the operator $T - \lambda I$ non-invertible?
c) What are the eigenvalues of $T$?
d) Explain the discrepancy between your answers to 2 and 3.

Proof. a) The operator $T + I$ is given by

$$(T + I)(a_1, a_2, a_3, \ldots) = (a_1 + 1, a_2 + 1, a_3 + 1, \ldots)$$

Let

$$S(a_1, a_2, a_3, \ldots) = (a_1 - a_1, a_2 - a_2 + a_1, a_3 - a_3 + a_2 - a_1, \ldots)$$

This is the inverse of $T + I$. To see this, we compute $S(T + I)(a_1, a_2, \ldots)$:

$$S(a_1, a_2 + 1, a_3 + 1, \ldots) = (a_1, a_2 + 1, a_3 + 1, \ldots)$$

Thus $S(T + I)v = v$ for all $v \in V$. We also need to compute $(T + I)S(a_1, a_2, \ldots)$:

$$(T + I)(a_1, a_2 - 1, a_3 - a_2 + 1, \ldots) = (a_1, (a_2 - 1) + a_1, (a_3 - a_2 + 1) + (a_2 - 1), \ldots)$$

Thus $(T + I)Sv = v$ for all $v \in V$. Since $S(T + I) = (T + I)S = I$, we get that $S = (T + I)^{-1}$.

b) The operator $T - \lambda I$ is given by

$$(T - \lambda I)(a_1, a_2, a_3, \ldots) = (-\lambda a_1, a_1 - \lambda a_2, a_2 - \lambda a_3, \ldots)$$

Let

$$S_\lambda = (-\frac{1}{\lambda}a_1, -\frac{1}{\lambda^2}a_2, \ldots)$$

For $\lambda \neq 0$, we will show that $S_\lambda = (T - \lambda I)^{-1}$. Indeed, if we write $S_\lambda(a_1, a_2, \ldots) = (b_1, b_2, \ldots)$ then the $n^{th}$ term of $S_\lambda$ is $b_n = -\frac{1}{\lambda}a_1 - \frac{1}{\lambda^2}a_2 - \cdots - \frac{1}{\lambda^n}a_n$, which is in fact $\frac{1}{\lambda}(b_{n-1} - a_n)$. We can see $b_1, b_2, \ldots$ all as functions from $V$ to $\mathbb{R}$.

We apply $S_\lambda$ to $(T - \lambda I)(a_1, a_2, \ldots)$. We have that $b_1(a_1, a_2, \ldots)$ is $-\frac{1}{\lambda}(-\lambda a_1) = a_1$. The $n^{th}$ term of $(T - \lambda I)(a_1, a_2, \ldots)$ is $a_{n-1} - \lambda a_n$. Suppose $b_{n-1}(T - \lambda I)(a_1, a_2, \ldots) = a_{n-1}$. Then

$$b_n(T - \lambda I)(a_1, a_2, \ldots) = \frac{1}{\lambda}(b_{n-1}(T - \lambda I)(a_1, a_2, \ldots) - (a_{n-1} - \lambda a_n))$$

(because $b_n(a_1, a_2, \ldots) = b_{n-1} - a_n$)

$$= \frac{1}{\lambda}(b_{n-1} - (a_{n-1} - \lambda a_n))$$

(since by assumption, $b_{n-1}(T - \lambda I)(a_1, \ldots) = a_n$)

$$= a_n$$

So by induction, $S_\lambda(T - \lambda I) = I$. This can be seen by direct computation for the first few terms:

$$S_\lambda(-\lambda a_1, a_1 - \lambda a_2, a_2 - \lambda a_3, \ldots) =$$

$$(a_1, -\frac{1}{\lambda^2}(-\lambda a_1) - \frac{1}{\lambda^3}(a_1 - \lambda a_2), -\frac{1}{\lambda^4}(a_1 - \lambda a_2) - \frac{1}{\lambda^5}(a_2 - \lambda a_3), \ldots)$$

$$= (a_1, a_2, a_3, \ldots)$$
Next, we must show that \((T - \lambda I)S_\lambda(a_1, a_2, \ldots) = (a_1, a_2, \ldots)\). Once again, we use that the \(n\)th term of \(S_\lambda(a_1, a_2, \ldots)\) is \(b_n = \frac{1}{\lambda}(b_{n-1} - a_n)\), and that the \(n\)th term of \((T - \lambda I)(a_1, a_2, \ldots)\) is \(a_{n-1} - \lambda a_n\). Thus,

\[
(T - \lambda I)(S_\lambda(a_1, a_2, \ldots)) = (T - \lambda I)(b_1, b_2, \ldots)
\]

\[
= (-\lambda b_1, b_1 - \lambda b_2, \ldots, b_{n-1} - \lambda b_n, \ldots)
\]

\[
= (-\lambda(\frac{1}{\lambda} a_1), b_1 - \lambda(\frac{1}{\lambda}(b_1 - a_n)), \ldots, b_{n-1} - \lambda(\frac{1}{\lambda}(b_{n-1} - a_n)), \ldots)
\]

\[
= (a_1, a_2, \ldots, a_n, \ldots)
\]

Therefore \((T - \lambda I)S_\lambda = I\). Since \((T - \lambda I)S_\lambda = S_\lambda(T - \lambda I) = I, S_\lambda = (T - \lambda I)^{-1}\) for all \(\lambda \neq 0\).

Thus \(T - \lambda I\) is invertible for all \(\lambda \neq 0\). However, for \(\lambda = 0\) we have \(T - \lambda I = T - 0I = T\), and \(T\) is not invertible. Indeed, the image of \(T\) is clearly contained in the subspace \(\{(0, *, *, *, \ldots)\}\) of sequences whose first entry is 0, so \(T\) is not surjective. Since \(T\) is not surjective, it cannot be bijective, so it cannot have an inverse even as a map of sets.

(However, note that if we let \(S\) be the backwards shift:

\[
S(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots)
\]

Then applying \(S_0\) to \(T\), we get

\[
S(T(a_1, a_2, \ldots)) = S(0, a_1, a_2, \ldots)
\]

\[
= (a_1, a_2, \ldots)
\]

So \(ST = I\), which might lead us to think that \(T\) is invertible.

However,

\[
T(S(a_1, a_2, a_3, \ldots)) = T(a_2, a_3, \ldots)
\]

\[
= (0, a_2, a_3, \ldots)
\]

So \(TS \neq I\), and so we see that \(S\) is not an inverse for \(T\).

So \(T - \lambda I\) is not invertible only for \(\lambda = 0\). \(\square\)

c) Suppose \(\lambda\) is an eigenvalue of \(T\). Then \(T(a_1, a_2, \ldots) = \lambda(a_1, a_2, \ldots)\) meaning that

\[
(0, a_1, a_2, \ldots) = (\lambda a_1, \lambda a_2, \ldots)
\]

This gives us that \(\lambda a_1 = 0\), so either \(\lambda = 0\) or \(a_1 = 0\). This equation also gives us \(\lambda a_n = a_{n-1}\) for \(n \geq 2\). If \(\lambda = 0\), then \(a_1, a_2, \ldots\) all equal zero. Thus \(\lambda\) is not an eigenvalue. If \(a_1 = 0\) but \(\lambda \neq 0\) then \(\lambda a_2 = a_1\) implies that \(a_2 = 0\), and so on. So if \(\lambda \neq 0\) then we also get that \(a_1 = a_2 = \cdots = 0\). Therefore \(T\) has no eigenvalues. \(\square\)

d) The discrepancy is that \(T - \lambda I\) is not invertible when \(\lambda = 0\), but 0 is not an eigenvalue of \(T\). In the finite-dimensional case, when an operator is not invertible, it is also not injective by Rank-Nullity. If \(T - \lambda I\) were not injective that would mean that \((T - \lambda I) \neq \{0\}\), so \(\lambda\) would be an eigenvalue. However, \(V\) is infinite-dimensional. In the infinite-dimensional case, an operator can be not invertible, and still be injective because Rank-Nullity no longer holds (nor does it make sense.) \(T\) is an example of such an operator that is injective but not invertible. That is why we have that \(T\) is not invertible, but 0 is not an eigenvalue of \(T\). \(\square\)