Homework 6
Due Wednesday, November 4 in class.

Do all the following exercises.

6A.16 over \( \mathbb{R} \)
6B.5  6B.7  6B.8

6A.16: you can assume that \( V \) is a vector space over \( \mathbb{R} \).
6B.5: uses the Gram-Schmidt algorithm, which we may not cover until Monday. If you want to start early, it’s on p183.
6B.7 and 6B.8: you may find these much easier after solving 6B.5.

Question 1. Let \( V \) be a finite-dimensional vector space over \( \mathbb{C} \), and let \( T \in \mathcal{L}(V) \). Let \( U \) and \( W \) be nonzero subspaces such that \( V = U \oplus W \).

Assume that \( U \) and \( W \) are invariant under \( T \), so we can restrict the operator \( T: V \to V \) to an operator \( T|_U: U \to U \), and similarly we can restrict \( T \) to an operator \( T|_W: W \to W \).

a) Prove without using minimal polynomials that if \( \lambda \in \mathbb{C} \) is an eigenvalue of \( T \), then either \( \lambda \) is an eigenvalue of \( T|_U \) or \( \lambda \) is an eigenvalue of \( T|_W \) (or both).

[Hint: start with a nonzero eigenvector \( v \in V \) such that \( T(v) = \lambda v \), and somehow construct either an eigenvector \( u \in U \) such that \( T(u) = \lambda u \), or an eigenvector \( w \in W \) such that \( T(w) = \lambda w \).]

Let \( f(x) \) be the minimal polynomial of \( T|_U \), and let \( g(x) \) be the minimal polynomial of \( T|_W \).

b) Prove that \( f(T)g(T) = 0 \) in \( \mathcal{L}(V) \).

c) Prove that if \( f(x) \) and \( g(x) \) have no shared roots (meaning no \( \lambda \in \mathbb{C} \) is a root of both \( f(x) \) and \( g(x) \)), then \( f(x)g(x) \) is the minimal polynomial of \( T \).

d) Prove that if \( f(x) \) and \( g(x) \) have a shared root \( \lambda \in \mathbb{C} \), then \( f(x)g(x) \) is not the minimal polynomial of \( T \).

Question 2. Let \( V \) be an inner product space over \( \mathbb{R} \), and suppose that \( T \in \mathcal{L}(V) \) satisfies \( \|Tv\| = \|v\| \) for all \( v \in V \). Prove that \( T \) has at most two eigenvalues.

(Question 3 provides an example showing that this does not hold for operators on inner product spaces over \( \mathbb{C} \).)

Question 3. Fix an integer \( n \geq 1 \), and let \( V = \mathbb{C}^n \) with the standard inner product. We let \( R: V \to V \) be the operator defined by

\[
R(a_1, \ldots, a_n) = (a_2, \ldots, a_n, a_1).
\]

(a) Set \( p(x) = x^n - 1 \). Prove that \( p(R) = 0 \).
(b*) Convince yourself that \( p(x) = x^n - 1 \) is in fact the minimal polynomial of \( R \). (Hint: choose a small \( n \), write the matrices for \( I, R, R^2, \ldots, R^{n-1} \) and see that they are linearly independent.) (You do not have to turn anything in for this part.)

This means that the eigenvalues of \( R \) are the roots of \( x^n - 1 \); since you might not be familiar with these awesome numbers (called “roots of unity”), here are the relevant facts.

Let \( \omega \in \mathbb{C} \) be the complex number \( \omega = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right) \). Then \( x^n - 1 \) factors as

\[
(x - 1)(x - \omega)(x - \omega^2) \cdots (x - \omega^{n-1})
\]

All the roots \( 1, \omega, \omega^2, \ldots, \omega^{n-1} \) are on the unit circle in \( \mathbb{C} \) (meaning \( z \bar{z} = 1 \)), and in fact they are equally spaced around the unit circle until you get back to \( \omega^n = 1 \).

(c) Since \( p(x) \) has \( n \) distinct roots, we know that \( R \) is diagonalizable.

Diagonalize \( R \) by finding a basis of eigenvectors \( v_1, \ldots, v_n \) for \( \mathbb{C}^n \) satisfying

\[
R(v_k) = \omega^k \cdot v_k \quad \text{and} \quad ||v_k|| = 1.
\]

(d) Prove that if \( \mu \in \mathbb{C} \) satisfies \( \mu^n = 1 \) but \( \mu \neq 1 \), then \( 1 + \mu + \mu^2 + \cdots + \mu^{n-1} = 0 \).

[Hint: multiply by \( \mu - 1 \).]

(e) Prove that your basis \( v_1, \ldots, v_n \) is orthonormal.

(f) If \( v = (a_1, \ldots, a_n) \) is written as \( v = b_1 v_1 + \cdots + b_n v_n \), give a formula for the coefficient \( b_i \) in terms of the coordinates \( a_1, \ldots, a_n \). [Hint: use part (e).]

(g) If \( v = (a_1, \ldots, a_n) \) is written as \( v = b_1 v_1 + \cdots + b_n v_n \), give a formula for the coordinate \( a_i \) in terms of the coefficients \( b_1, \ldots, b_n \). [Hint: this is easy.]

(h) If \( v = (a_1, \ldots, a_n) \) is written as \( v = b_1 v_1 + \cdots + b_n v_n \), prove that the coordinates \( a_1, \ldots, a_n \) and the coefficients \( b_1, \ldots, b_n \) satisfy the relation

\[
|a_1|^2 + \cdots + |a_n|^2 = |b_1|^2 + \cdots + |b_n|^2.
\]

**Remark:** The formula you found in (f) is the Fourier transform, or rather a discretized version of it; the formula you found in (g) is the inverse Fourier transform.

The equality you proved in (h) is a discrete version of the famous Plancherel theorem (also known as Rayleigh’s energy theorem): If \( f : [-\pi, \pi] \rightarrow \mathbb{C} \) is a continuous function with \( f(-\pi) = f(\pi) \), we saw in class that

\[
\text{energy}(f)^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx.
\]

Let the Fourier coefficients \( b_k \in \mathbb{C} \) be the sequence defined for \( k \in \mathbb{Z} \) by

\[
b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx} \, dx.
\]

Then energy\((f)^2 \) can be computed by either side of:

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{k=-\infty}^{\infty} |b_k|^2.
\]