Math 113 Homework 4 Solutions

Solutions by Guanyang Wang, with edits by Tom Church.

Exercises from the book.

Exercise 3.E.13 Suppose $U$ is a subspace of $V$ and $v_1 + U, \ldots, v_m + U$ is a basis of $V/U$ and $u_1, \ldots, u_n$ is a basis of $U$. Prove that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of $V$.

Proof. First we prove that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a linearly independent list in $V$.

Suppose that $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$ are scalars such that

$$a_1 v_1 + \ldots + a_m v_m + b_1 u_1 + \ldots + b_n u_n = 0$$

Thus $a_1 v_1 + \ldots + a_m v_m = -(b_1 u_1 + \ldots + b_n u_n) \in U$, which implies that

$$a_1 (v_1 + U) + \ldots + a_m (v_m + U) = 0 + U$$

Because $v_1 + U, \ldots, v_m + U$ is a basis of $V/U$, this implies $a_1 = \ldots = a_m = 0$. Therefore we have

$$b_1 u_1 + \ldots + b_n u_n = 0$$

Since $u_1, \ldots, u_n$ is a basis of $V$, this implies $b_1 = \ldots = b_n = 0$. Thus $a_1 = \ldots = a_m = b_1 = \ldots = b_n = 0$, which implies that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a linearly independent list in $V$.

Then we prove that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of $V$. Now suppose $v \in V$. Because the list $v_1 + U, \ldots, v_m + U$ spans $V/U$, there exist $c_1, \ldots, c_m \in \mathbb{F}$ such that

$$v + U = c_1 (v_1 + U) + \ldots + c_m (v_m + U)$$

Thus

$$v - c_1 v_1 - \ldots - c_m v_m \in U$$

Because the list $u_1, \ldots, u_n$ spans $V/U$, there exist $d_1, \ldots, d_n \in \mathbb{F}$ such that

$$v - c_1 v_1 - \ldots - c_m v_m = d_1 u_1 + \ldots + d_n u_n.$$ 

Hence

$$v = c_1 v_1 + \ldots + c_m v_m + d_1 u_1 + \ldots + d_n u_n.$$ 

Then the list $v_1, \ldots, v_m, u_1, \ldots, u_n$ spans $V$ and hence is a basis of $V$, as desired. \qed

Exercise 3.F.7 Suppose $m$ is a positive integer. Show that the dual basis of the basis $1, x, \ldots, x^n$ of $\mathcal{P}_m(\mathbb{R})$ is $\varphi_0, \varphi_1, \ldots, \varphi_m$, where $\varphi_j(p) = \frac{p^{(j)}(0)}{j!}$. Here $p^{(j)}$ denotes the $j^{th}$ derivative of $p$, with the understanding that the $0^{th}$ derivative of $p$ is $p$.

Proof. From Proposition 3.98 we know that the dual basis is a basis of dual space. By definition of dual basis (3.96), we just need to check if

$$(0.1) \quad \varphi_j(x^k) = \begin{cases} 1 & (j = k) \\ 0 & (j \neq k) \end{cases}$$

Note that $\varphi_j(x^k) = \frac{(x^k)^{(j)}(0)}{j!}$, hence if $j = k$, $\varphi_j(x^k) = 1$, if $j \neq k$, $\varphi_j(x^k) = 0$. Therefore we know that $\varphi_0, \ldots, \varphi_m$ is the dual basis of $\mathcal{P}_m(\mathbb{R})$. 

Exercise 3.F.8 Suppose $m$ is a positive integer.
(a) Show that $1, x-5, \ldots, (x-5)^m$ is a basis of $P_m(\mathbb{R})$.
(b) What is the dual basis of the basis in part (a)?

Proof. (a) Define $\varphi_0, \varphi_1, \ldots, \varphi_m \in (P_m(\mathbb{R}))'$ by

$$
\varphi_j(p) = \frac{p^{(j)}(5)}{j!}.
$$

So suppose $a_0, \ldots, a_m \in F$ and

$$
a_0 + a_1(x-5) + \ldots + a_m(x-5)^m = 0.
$$

Then for $j = 0, 1, \ldots, m$, we have

$$
a_j = \varphi_j(a_0 + a_1(x-5) + \ldots + a_m(x-5)^m) = \varphi_j(0) = 0.
$$

Thus $a_0 = a_1 = \ldots = a_m = 0$. Hence $1, x-5, \ldots, (x-5)^m$ is a linearly independent list in $P_m(\mathbb{R})$ of length $m+1$, which equals the dimension of $P_m(\mathbb{R})$. Thus $1, x-5, \ldots, (x-5)^m$ is a basis of $P_m(\mathbb{R})$ (by 2.39).

(b) Let $\varphi_0, \varphi_1, \ldots, \varphi_m \in (P_m(\mathbb{R}))'$ be defined as in part (a). Then we have

$$
\varphi_j((x-5)^k) = \begin{cases} 
1 & (j = k) \\
0 & (j \neq k)
\end{cases}
$$

From Proposition 3.98 we know that $\varphi_0, \varphi_1, \ldots, \varphi_m$ is the dual basis of the basis in part (a). □

Exercise 3.F.15 Suppose $W$ is finite-dimensional and $T \in \mathcal{L}(V,W)$. Prove that $T' = 0$ if and only if $T = 0$.

Proof. First suppose $T = 0$. For any $\varphi \in W'$, then $T'((\varphi)) = \varphi \circ T = 0$, and thus $T' = 0$.

To prove the other direction, now suppose $T' = 0$. Thus

$$
0 = T'((\varphi)) = \varphi \circ T
$$

for every $\varphi \in W'$.

If $T \neq 0$, we can find some $v \in V$ such that $Tv = w \neq 0$. We can extend $Tv$ to a basis $Tv, w_2, \ldots, w_n$ of $W$. Now Proposition 3.5 implies that there exists a $\tilde{\varphi}$ such that $\tilde{\varphi}(Tv) = 1$ (and $\tilde{\varphi}(v_j)$ equals whatever we want for $j = 2, 3, \ldots, n$). Therefore $(T'(\tilde{\varphi}))(v) = \tilde{\varphi}(Tv) = 1$. Which contradicts the fact that $0 = T'((\varphi)) = \varphi \circ T$ for every $\varphi \in W'$. So we must have $T = 0$, as desired. □

Exercise 5.A.12 Define $T \in \mathcal{L}(P_4(\mathbb{R}))$ by

$$
(Tp)(x) = xp'(x)
$$

for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors of $T$. 

□
Answer. A typical element \( p \) of \( \mathcal{P}_4(\mathbb{R}) \) is given by expression
\[
p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4,
\]
where \( a_0, \ldots, a_4 \in \mathbb{R} \).

With that expression, the eigenvalue-eigenvector equation \( Tp = \lambda p \), which in this case is \( xp'(x) = \lambda p(x) \), becomes
\[
a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 = \lambda (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)
\]

Comparing coefficients in the equation above, we see that the eigenvalue-eigenvector equation is equivalent to the system of equations
\[
0 = \lambda a_0 \\
a_1 = \lambda a_1 \\
2a_2 = \lambda a_2 \\
3a_3 = \lambda a_3 \\
4a_4 = \lambda a_4.
\]

From the equations above, we can see that if \( j \in \{0, 1, 2, 3, 4\} \) and \( a_j \neq 0 \), then we have \( \lambda = j \) and \( a_k = 0 \) for any \( k \neq j \). Thus the eigenvalue of \( T \) are \( 0, 1, 2, 3, 4 \) and the corresponding eigenvectors are of the form \( c, cx, cx^2, cx^3, cx^4 \), where \( c \in \mathbb{R} \) and \( c \neq 0 \).

Exercise 5.A.15 Suppose \( T \in \mathcal{L}(V) \). Suppose \( S \in \mathcal{L}(V) \) is invertible
(a) Prove that \( T \) and \( S^{-1}TS \) have the same eigenvalues.
(b) What is the relationship between the eigenvectors of \( T \) and the eigenvectors of \( S^{-1}TS \)?

Answer. Suppose \( v \in V \) and \( \lambda \in \mathbb{F} \). Then we have
\[
Tv = \lambda v \iff (S^{-1}TS)(S^{-1}v) = \lambda S^{-1}v
\]
This is because if \( Tv = \lambda v \), then \( (S^{-1}TS)(S^{-1}v) = S^{-1}Tv = \lambda S^{-1}v \), on the other hand, if \( (S^{-1}TS)(S^{-1}v) = \lambda S^{-1}v \), then \( \lambda v = \lambda S(S^{-1}v) = S((S^{-1}TS)(S^{-1}v)) = Tv \).

Thus we see that \( T \) and \( S^{-1}TS \) have the same eigenvalues, and furthermore, \( v \) is an eigenvector of \( T \) if and only if \( S^{-1}v \) is an eigenvector of \( S^{-1}TS \).

Exercise 5.A.18 Show that the operator \( T \in \mathcal{L}(\mathbb{C}^\infty) \) defined by
\[
T(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)
\]
has no eigenvalues.

Answer. The eigenvalue-eigenvector equation \( Tz = \lambda z \) for this operator is
\[
(0, z_1, z_2, \ldots) = (\lambda z_1, \lambda z_2, \lambda z_3, \ldots)
\]
which is equivalent to
\[
0 = \lambda z_1, z_1 = \lambda z_2, z_2 = \lambda z_3, \ldots
\]
The first equation implies \( z_1 = 0 \) or \( \lambda = 0 \). If \( \lambda = 0 \), then the rest of the equations implies \( 0 = z_1 = z_2 = \ldots \), which eliminates 0 as the possible eigenvalue. If \( \lambda \neq 0 \), then \( z_1 = 0 \), then the rest of the equations also implies \( z_2 = z_3 = \ldots = 0 = z_1 \).
which eliminates all nonzero complex numbers \( \lambda \) as possible eigenvalues. Thus we conclude that \( T \) has no eigenvalues. \( \square \)

**Exercise 5.A.20** Find all eigenvalues and eigenvectors of the backward shift operator \( T \in \mathcal{L}(F^\infty) \) defined by

\[
T(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots)
\]

**Answer.** We will show that all \( \lambda \in F \) are eigenvalues of \( T \), and the set of eigenvectors of \( T \) with eigenvalue \( \lambda \) is the set \( V_\lambda = \{(z, \lambda z, \lambda^2 z, \ldots) \mid z \in F\} \).

First we show that if \( v \) is an eigenvector of \( T \), then \( v \in V_\lambda \) for some \( \lambda \). That is, we show that \( v = (z, \lambda z, \lambda^2 z, \ldots) \) for some \( z \) and some \( \lambda \). Suppose \( v = (z_1, z_2, z_3, \ldots) \) is an eigenvector for \( T \) with eigenvalue \( \lambda \). Then the eigenvalue equation \( T(v) = \lambda v \) takes the form

\[
(\lambda z_1, \lambda z_2, \lambda z_3, \ldots) = (z_2, z_3, z_4, \ldots)
\]

Since two vectors in \( F^\infty \) are equal if and only if their terms are all equal, this yields an infinite sequence of equations:

\[
z_2 = \lambda z_1, \quad z_3 = \lambda z_2, \ldots, \quad z_n = \lambda z_{n-1}, \ldots
\]

From this, we can repeatedly substitute \( z_n = \lambda z_{n-1} = \lambda^2 z_{n-2} = \ldots \), so in fact (by a simple induction)

\[
z_n = \lambda^{n-1} z_1
\]

So every eigenvector \( v \) with eigenvalue \( \lambda \) is of the form \( v = (z_1, \lambda z_1, \lambda^2 z_1, \ldots) \). Furthermore, for any \( z \in F \), if we set \( z_1 = z \), \( z_2 = \lambda z \), \ldots, \( z_n = \lambda^n z \), the vector

\[
v = (z, \lambda z, \lambda^2 z, \ldots)
\]

satisfies the equations above and is an eigenvector of \( T \) with eigenvalue \( \lambda \). Therefore, the eigenspace \( V_\lambda \) of \( T \) with eigenvalue \( \lambda \) is the set of vectors

\[
V_\lambda = \{(z, \lambda z, \lambda^2 z, \ldots) \mid z \in F \}.
\]

Finally, we show that every single \( \lambda \in F \) occurs as an eigenvalue of \( T \). Given \( \lambda \in F \), consider the vector \( v = (1, \lambda, \lambda^2, \ldots) \). Applying \( T \) to \( v \), we get

\[
T(v) = (1, \lambda, \lambda^2, \ldots) = (\lambda, \lambda^2, \lambda^3, \ldots) = \lambda(1, \lambda, \lambda^2, \ldots)
\]

Thus \( T(v) = \lambda v \) for this vector. We have thus shown that all \( \lambda \in F \) are eigenvalues for \( T \), and the eigenspace for \( \lambda \) is \( V_\lambda = \{(z, \lambda z, \lambda^2 z, \ldots) \mid z \in F \} \). \( \square \)

**Exercise 5.A.22** Suppose \( T \in \mathcal{L}(V) \) and there exist nonzero vectors \( v \) and \( w \) in \( V \) such that

\[
Tv = 3w \quad \text{and} \quad Tw = 3v.
\]

Prove that 3 or \(-3\) is an eigenvalue of \( T \).

**Proof.** The equations above imply that

\[
T(v + w) = 3(v + w) \quad \text{and} \quad T(v - w) = -3(v - w).
\]

The vectors \( v + w \) and \( v - w \) cannot both be 0 (because otherwise we would have \( v = w = 0 \)). Thus the equations above imply that 3 or \(-3\) is an eigenvalue of \( T \). \( \square \)
Exercise 5.A.30 Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ and $4, -5$ and $\sqrt{7}$ are the eigenvalues of $T$. Prove that there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (4, -5, \sqrt{7})$.

Proof. Since $T$ has at most 3 distinct eigenvalues (by 5.13), the hypothesis imply that $9$ is not an eigenvalue of $T$. Thus $T - 9I$ is surjective. In particular, there exists $x \in \mathbb{R}^3$ such that $(T - 9I)x = Tx - 9x = (4, -5, \sqrt{7})$. (The entries of this particular vector are a red herring: we could just as easily find a $y \in \mathbb{R}^3$ such that $Ty - 9y = (86, 75, 309)$ by the same argument.) □

Exercise 5.A.32 Suppose $\lambda_1, ..., \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, ..., e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbb{R}$.

Proof. Let $V = \text{span}(e^{\lambda_1 x}, ..., e^{\lambda_n x})$, and define $T \in \mathcal{L}(V)$ by $Tf = f'$. This linear map does map $V$ into $V$ because $T(e^{\lambda_j x}) = \lambda_j e^{\lambda_j x}$.

This equation above also shows that for each $j = 1, ..., n$, the vector $e^{\lambda_j x}$ is an eigenvector of $T$ with eigenvalue $\lambda_j$. Thus Proposition 5.10 implies that $e^{\lambda_1 x}, ..., e^{\lambda_n x}$ is linearly independent. □

Exercise 5.B.1 Suppose $T \in \mathcal{L}(V)$ and there exists a positive integer $n$ such that $T^n = 0$.

(a) Prove that $I - T$ is invertible and that

$$(I - T)^{-1} = I + T + ... + T^{n-1}$$

(b) Explain how you would guess the formula above.

Proof. We have

$$(I - T)(I + T + ... + T^{n-1})$$

$$= I + T + ... + T^{n-1} - T - T^2 - ... - T^{n-1} - T^n$$

$$= I - T^n = I$$

since $T^n = 0$.

Similarly, we have

$$(I + T + ... + T^{n-1})(I - T)$$

$$= I + T + ... + T^{n-1} - T - T^2 - ... - T^{n-1} - T^n$$

$$= I - T^n = I$$

Therefore $(I - T)$ is invertible and $(I - T)^{-1} = I + T + ... + T^{n-1}$.

(b) If $r \in \mathbb{C}$ and $|r| < 1$, then we might be familiar with the usual formula for the sum of a geometric series:

$$(1 - r)^{-1} = 1 + r + r^2 + ... + r^n + r^{n+1} + ...$$

If we guess that in the formula above, we can replace 1 with $I$ and $r$ with $T$, then we would have

$$(I - T)^{-1} = I + T + ... + T^{n-1}$$
where the sum becomes finite because $0 = T^n = T^{n+1} = \cdots$.

**Exercise 5.B.2** Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose $\lambda$ is an eigenvalue of $T$. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

**Proof.** Let $v \in V$ be a eigenvector of $T$ corresponding to the eigenvalue $\lambda$. Then

$$0 = (T - 2I)(T - 3I)(T - 4I)v = (\lambda - 2)(\lambda - 3)(\lambda - 4)v.$$

Since $v \neq 0$, the equation above implies that

$$(\lambda - 2)(\lambda - 3)(\lambda - 4) = 0$$

Thus $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$, as desired. □

**Question 1.** Suppose $U$ is a subspace of $V$ such that $\dim V/U = 1$. Prove that there exists a linear functional $f \in V'$ such that

$$\text{null } f = U$$

**Proof.** Since $V/U$ is a 1-dimensional linear space, we can construct an arbitrary nonzero linear map $g \in (V/U)'$. Definition 3.88 says we have a quotient map $\pi: V \to V/U$ which sends $v \in V$ to $v + U \in V/U$. Now let $f = g \circ \pi$. We claim that $\text{null } f = U$

On the one hand, for any $u \in U$, $\pi(u) = 0 + U = 0$ in $V/U$, so we have $f(u) = g(\pi(u)) = g(0) = 0$, therefore $\text{null } f \supset U$.

On the other hand, since $g \neq 0$, we can find $v + U \in V/U$ such that $g(v + U) \neq 0$, so $f(v) = g(\pi(v)) = g(v + U) \neq 0$. Since $\dim V/U = 1$, $v + U$ is the basis of $V/U$. Therefore for any $w \not\in U$, we can find a non-zero $\lambda \in \mathbb{F}$ and such that $\pi(w) = w + U = \lambda(v + U) = \lambda v + U$. So we have

$$f(w) = g(\pi(w)) = g(w + U) = g(\lambda(v + U)) = \lambda g(v + U) \neq 0$$

because $\lambda \neq 0$ and $g(v + U) \neq 0$. So $\text{null } f \subset U$.

Hence we have proved that $\text{null } f = U$, as desired. □

**Question 2.** Let $C^\infty(\mathbb{R})$ denote the vector space (over $\mathbb{R}$) of infinitely-differentiable real-valued functions $f: \mathbb{R} \to \mathbb{R}$.

a) Let $U$ denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which vanish at 42 and at $\pi$:

$$U = \{ f \in C^\infty(\mathbb{R}) \mid f(42) = 0, f(\pi) = 0 \}$$

Prove that the quotient vector space $C^\infty(\mathbb{R})/U$ is finite dimensional. What is its dimension?
b) Let \( W \) denote the subspace of \( C^\infty(\mathbb{R}) \) consisting of functions which “vanish to second order at 0”:

\[
W = \{ f \in C^\infty(\mathbb{R}) \mid f(0) = 0, f'(0) = 0, f''(0) = 0 \}
\]

Prove that the quotient vector space \( C^\infty(\mathbb{R})/W \) is finite dimensional, and find a basis for \( C^\infty(\mathbb{R})/W \).

**Proof.** a) Define the linear transformation \( T : C^\infty(\mathbb{R}) \to \mathbb{R}^2 \) by

\[
T(f) = (f(42), f(\pi)).
\]

The kernel of \( T \) is

\[
\ker T = \{ f \in C^\infty(\mathbb{R}) \mid T(f) = 0 \}
= \{ f \in C^\infty(\mathbb{R}) \mid f(2) = 0, f(7) = 0 \}
= U.
\]

The Quotient Isomorphism Theorem (Thm 3.91(d)) thus tells us that \( T : C^\infty(\mathbb{R})/U \to \) \( \text{Image } T \) is an isomorphism, so we need to understand \( \text{Image } T \).

Choose two functions \( f, g \in C^\infty(\mathbb{R}) \) that satisfy \( T(f) = (1, 0) \) and \( T(g) = (0, 1) \), such as:

\[
f(x) = \frac{x - \pi}{42 - \pi},
g(x) = \frac{42 - x}{42 - \pi},
\]

Since

\[
f(42) = 1 \quad f(\pi) = 0
\]
\[
g(42) = 0 \quad g(\pi) = 1
\]

we have \( T(f) = (1, 0) \) and \( T(g) = (0, 1) \). This shows that \( (1, 0) \in \text{Image } T \) and \( (0, 1) \in \text{Image } T \). Since these are the standard basis vectors \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \), they span \( \mathbb{R}^2 \), and so \( \text{Image } T = \mathbb{R}^2 \).

Since \( \text{Image } T = \mathbb{R}^2 \), the Quotient Isomorphism Theorem (Thm 3.91(d)) states that \( T : C^\infty(\mathbb{R})/U \to \mathbb{R}^2 \) is an isomorphism. Since \( C^\infty(\mathbb{R})/U \) and \( \mathbb{R}^2 \) are isomorphic, they have the same dimension: therefore \( C^\infty(\mathbb{R})/U \) has dimension 2.

b) Define the linear transformation \( S : C^\infty(\mathbb{R}) \to \mathbb{R}^3 \) by

\[
S(f) = (f(0), f'(0), f''(0)).
\]

The kernel of \( S \) is

\[
\ker S = \{ f \in C^\infty(\mathbb{R}) \mid S(f) = 0 \}
= \{ f \in C^\infty(\mathbb{R}) \mid f(0) = 0, f'(0) = 0, f''(0) = 0 \}
= W.
\]

The Quotient Isomorphism Theorem (Thm 3.91(d)) thus tells us that \( S : C^\infty(\mathbb{R})/W \to \) \( \text{Image } S \) is an isomorphism, so we need to understand \( \text{Image } S \). Consider the following functions in \( C^\infty(\mathbb{R}) \):

\[
f_1 = 1
\]
\[
f_2 = x - 1
\]

---

1 For example, if \( f(x) = x \) then \( T(f) = (42, \pi) \); if \( g(x) = e^x \) then \( T(g) = (e^{42}, e^\pi) \), if \( h(x) = \sin x \) then \( T(h) = (\sin 42, \sin \pi) \), etc.

2 Many other choices are possible.

3 For example, if \( f(x) = x^2 \) then \( S(f) = (0, 0, 2) \); if \( g(x) = e^x \) then \( S(g) = (1, 1, 1) \); if \( h(x) = \sin x \) then \( S(h) = (0, 1, 0) \), etc.
\[ f_3 = x^2 - 2x + 1 \]

These three functions are infinitely differentiable, so they are in \( C^\infty(\mathbb{R}) \). Their only important properties are that

\[
\begin{align*}
  f_1(0) &= 1 & f_1'(0) &= 0 & f_1''(0) &= 0 \\
  f_2(0) &= 0 & f_2'(0) &= 1 & f_2''(0) &= 0 \\
  f_3(0) &= 0 & f_3'(0) &= 0 & f_3''(0) &= 1
\end{align*}
\]

This implies that

\[ S(f_1) = e_1, \quad S(f_2) = e_2, \quad S(f_3) = e_3. \]

Therefore \( e_1, e_2, \) and \( e_3 \) are all in \( \text{Image}S \). Since \( e_1, e_2, e_3 \) is a basis for \( \mathbb{R}^3 \), this shows that \( \text{Image}S = \mathbb{R}^3 \).

Since \( \text{Image}S = \mathbb{R}^3 \), the Quotient Isomorphism Theorem (Thm 3.91(d)) states that \( \overline{S} : C^\infty(\mathbb{R})/W \to \mathbb{R}^3 \) is an isomorphism. Since \( C^\infty(\mathbb{R})/W \) and \( \mathbb{R}^3 \) are isomorphic, they have the same dimension: therefore \( C^\infty(\mathbb{R})/W \) has dimension 3.

Consider the elements \( v_1 = f_1 + W, \ v_2 = f_2 + W, \) and \( v_3 = f_3 + W \) in the quotient space \( C^\infty(\mathbb{R})/W \). We will show they are linearly independent. Assume that \( av_1 + bv_2 + cv_3 = 0 \) in \( C^\infty(\mathbb{R})/W \). The above formula shows that

\[ \overline{S}(v_1) = \overline{S}(f_1 + W) = e_1, \quad \overline{S}(v_2) = \overline{S}(f_2 + W) = e_2, \quad \overline{S}(v_3) = \overline{S}(f_3 + W) = e_3. \]

Since \( \overline{S} \) is linear, \( \overline{S}(av_1 + bv_2 + cv_3) = ae_1 + be_2 + ce_3 \). But \( e_1, e_2, e_3 \) are linearly independent, so we conclude that \( a = b = c = 0 \). This shows that \( v_1, v_2, v_3 \) are linearly independent in the quotient space \( C^\infty(\mathbb{R})/W \). Since this vector space has dimension 3, this implies that \( v_1, v_2, v_3 \) is a basis for \( C^\infty(\mathbb{R})/W \).

**Question 3.** Let \( C^\infty(\mathbb{R}, \mathbb{C}) \) be the vector space (over \( \mathbb{C} \)) of complex-valued functions \( f : \mathbb{R} \to \mathbb{C} \) that are infinitely differentiable. Let \( V \) be the space of functions \( f \in C^\infty(\mathbb{R}, \mathbb{C}) \) satisfying the equation \( f'' = -f \):

\[ V = \{ f \in C^\infty(\mathbb{R}, \mathbb{C}) \mid f'' = -f \} \]

- Assume without proof that \( \dim V \leq 2 \). Prove that the functions \( \sin x \) and \( \cos x \) both lie in \( V \), and moreover that \( (\sin x, \cos x) \) form a basis for \( V \).
- Let \( D \) be the operator on \( C^\infty(\mathbb{R}, \mathbb{C}) \) defined by \( D(f) = f' \). Prove that \( V \) is an invariant subspace for \( D \).
- Now consider \( D \in L(V) \) as an operator on \( V \). Find a basis for \( V \) consisting of eigenvectors for \( D \). What are their eigenvalues?

**Proof.**

- Consider the functions \( \sin x \) and \( \cos x \). Then \( \sin''(x) = (\cos'(x)) = -\sin(x) \) and \( \cos''(x) = (-\sin(x))' = -\cos(x) \). Thus \( \sin(x), \cos(x) \in V \).

  To show that \( (\sin x, \cos x) \) form a basis for \( V \), first we show that they are linearly independent. Suppose there are numbers \( a, b \in \mathbb{C} \) s.t. \( a \sin(x) + b \cos(x) = 0 \). Then, plugging in \( x = 0 \), we get \( b = 0 \) since \( \sin(0) = 0 \) and \( \cos(0) = 1 \). Plugging in \( x = \pi/2 \), we get \( a = 0 \) since \( \sin(\pi/2) = 1 \) and \( \cos(\pi/2) = 0 \). Thus \( \sin(x) \) and \( \cos(x) \) are linearly independent.

  Since \( \sin(x) \) and \( \cos(x) \) are linearly independent, the dimension of \( V \) must be at least 2. Since we were given that \( \dim V \) is at most 2, we conclude that \( \dim V = 2 \). Thus \( \sin(x) \) and \( \cos(x) \) form a basis for \( V \).

- Let \( D \) be the operator on \( C^\infty(\mathbb{R}, \mathbb{C}) \) defined by \( D(f) = f' \). To show that \( V \) is invariant under \( D \), we must show that if \( f \in V \) then \( Df \in V \). So suppose that \( f \in V \), and set \( g = D(f) \). Then \( f'' = -f \). Differentiating both sides
of this equation, we get that $f''' = -f'$, or in other words $g'' = -g$. Thus $g = D(f)$ lies in $V$. Therefore, $V$ is invariant under $D$.

- Now consider $D \in \mathcal{L}(V)$ as an operator on $V$. Find a basis for $V$ consisting of eigenvectors for $D$. What are their eigenvalues?

  The properties $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$ mean that
  \[
  D(a \sin(x) + b \cos(x)) = -b \sin(x) + a \cos(x).
  \]
  We have seen a similar linear transformation in class, namely $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(x, y) = (-y, x)$.

  However that operator has no eigenvalues because it is on a real vector space, and its minimal polynomial $p(x) = x^2 + 1$ has no real roots. In contrast, here we are working over the complex numbers, so we might imagine that the eigenvalues would be the complex roots of $p(x) = x^2 + 1$, namely $i$ and $-i$.

  The eigenvalue equation $D(a \sin(x) + b \cos(x)) = i(a \sin(x) + b \cos(x))$ can be solved to find
  \[
  f = \cos(x) + i \sin(x)
  \]
  and similarly $D(a \sin(x) + b \cos(x)) = -i(a \sin(x) + b \cos(x))$ can be solved to find
  \[
  g = \cos(x) - i \sin(x).
  \]
  Then we can check that
  \[
  D(f) = -\sin(x) + i \cos(x) = if
  \]
  and
  \[
  D(g) = -\sin(x) - i \cos(x) = -ig.
  \]
  Thus $f$ and $g$ are eigenvectors for $D$ with eigenvalues $i$ and $-i$. Since they have distinct eigenvalues, Theorem 5.6 in the book implies that they are linearly independent. Since $\dim V \leq 2$, any spanning list of length 2 forms a basis for $V$.

  **Remark by TC:** you have probably learned what the eigenvectors of $D$ as an operator on $\mathbb{C}^\infty(\mathbb{R}, \mathbb{C})$ are in a previous class. For the eigenvalue $a$, the eigenvalue equation $D(f) = af$ becomes the differential equation $f' = af$, and you may already know that the solutions to this equation are (constant multiples of)
  \[
  f(x) = e^{ax}
  \]
  since the chain rule implies that
  \[
  (e^{ax})' = a \cdot e^{ax}.
  \]
  But the functions $f$ and $g$ you found above are eigenvectors with eigenvalues $a = i$ and $a = -i$, so they must be of the form $Ce^{ix}$ and $Ce^{-ix}$. We can find the constants by plugging in 0, since $Ce^{i\cdot 0} = C$. By plugging in $f(0) = \cos(0) + i \sin 0 = 1 + i \cdot 0 = 1$ and $g(0) = \cos(0) - i \sin 0 = 1 - i \cdot 0 = 1$ we see that the constants are 1 for both $f$ and $g$. Therefore you have proved the famous formula of Euler:
  \[
  e^{ix} = \cos x + i \cdot \sin x \quad e^{-ix} = \cos x - i \cdot \sin x.
  \]
In particular, if we evaluate the first eigenfunction at $\pi$ we get
\[ e^{i\pi} = \cos \pi + i \cdot \sin \pi = -1 + i \cdot 0, \]
or in other words
\[ e^{i\pi} = -1. \]