Homework 3
Due Wednesday, October 14 in class.

Do all the following exercises, but write up only the unstarred exercises and the questions below. (Starred exercises are valuable and worth working out, but they will not be collected or graded.)

\[
\begin{array}{cccc}
3B.2^* & 3B.12^* & 3B.20 & 3B.21^* & 3B.29 \\
3C.3 & 3C.4^* & 3C.5^* & 3D.7 & 3D.9^* & 3D.10^* & 3D.16
\end{array}
\]

**Question 1.** Let $T \in \mathcal{L}(V)$ be an operator on $V$. Recall that $T^2$ denotes the composition $T \circ T$.

a) Give an example of a vector space $V$ and an operator $T \in \mathcal{L}(V)$ such that $T^2 = T$ (an example other than $T = I$ or $T = 0$, those are too easy).

b) Prove that if $T^2 = T$, then $V = \text{null} T \oplus \text{null}(T - I)$.

c) Prove that if $V = \text{null} T + \text{null}(T - I)$, then $T^2 = T$.

d) Give an example of a vector space $V$ and an operator $T \in \mathcal{L}(V)$ such that $T^2 = -I$.

**Question 2.** Let $V$ and $W$ be finite dimensional, and consider $T \in \mathcal{L}(V,W)$ and $S \in \mathcal{L}(W,U)$.

a) Prove that $\dim(\text{range } ST) \leq \dim(\text{range } T)$.

b) Prove that $\dim(\text{range } ST) = \dim(\text{range } T)$ if and only if

\[ \text{range } T + \text{null } S = \text{range } T \oplus \text{null } S. \]

c) Prove that $\dim(\text{null } ST) \leq \dim(\text{null } S) + \dim(\text{null } T)$.

d) Challenge problem: Can you give some description (in terms of conditions on $T$, $S$, $V$, etc.) of when we get equality in the previous part, i.e. $\dim(\text{null } ST) = \dim(\text{null } S) + \dim(\text{null } T)$?