Math 113: Linear Algebra and Matrix Theory
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Homework 1
Due Wednesday, September 30 in class.

Do all the following exercises:
1A.2 1A.3 1A.10
1B.1 1B.2
1C.4 1C.20 1C.24

You may assume the following facts from calculus without proof:
- Sums of continuous functions are continuous.
- A scalar multiple of a continuous function is continuous.
- Sums of differentiable functions are differentiable.
- A scalar multiple of a differentiable function is differentiable.

**Question 1.** Let $S$ be a set, and let $U$ be a vector space over $F$. Recall that $U^S$ is the set of functions $f: S \to U$. Given functions $f, g \in U^S$ and $a \in F$, we define $f + g \in U^S$ and $a \cdot f \in U^S$ by

$$
(f + g)(x) = f(x) + g(x),
(a \cdot f)(x) = a \cdot (f(x))
$$

Prove that $U^S$ is a vector space over $F$.

**Question 2.** Let $U_1 = \{(a, 0, 0) \mid a \in F\}$ and $U_2 = \{(b, b, 0) \mid b \in F\}$. These are both subsets of $F^3$.

a) Prove that $U_1$ and $U_2$ are subspaces of $F^3$.

b) Prove that $U_1 + U_2 = \{(x, y, 0) \mid x, y \in F\}$.

**Question 3.** Let $V$ be a vector space, and let $U_1$ and $U_2$ be subspaces of $V$.

a) Their intersection $U_1 \cap U_2$ consists of all vectors that belong to both subspaces:

$$
U_1 \cap U_2 = \{v \in V \mid v \in U_1 \text{ and } v \in U_2\}.
$$

Prove that $U_1 \cap U_2$ is always a subspace of $V$.

b) Their union $U_1 \cup U_2$ consists of all vectors that belong to either subspace:

$$
U_1 \cup U_2 = \{v \in V \mid v \in U_1 \text{ or } v \in U_2\}.
$$

Prove that $U_1 \cup U_2$ is a subspace of $V$ if and only if one subspace is contained in the other.\[1\]

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\[1\] i.e. either $U_1 \subset U_2$ or $U_2 \subset U_1$. Notice that this means that the union of two subspaces is usually not a subspace.
Question 4. Let $U_1 = \{(a, -a, 0) \mid a \in F\}$, let $U_2 = \{(0, b, -b) \mid b \in F\}$, and let $U_3 = \{(c, 0, -c) \mid c \in F\}$. These are all subspaces of $F^3$ (you may assume this without proof).

a) Describe the subspace $U_1 + U_2 + U_3$ by filling in the blank by an equation involving $x$, $y$, and $z$:

$$U_1 + U_2 + U_3 = \{(x, y, z) \in F^3 \mid \text{___________} \}$$

b) Let $W = U_1 + U_2 + U_3$. Is $W$ the direct sum of $U_1$, $U_2$, and $U_3$? Prove or disprove.

Question 5. Let $U$ be the following subset of $F^\infty$:

$$U = \{(v_1, v_2, v_3, \ldots) \in F^\infty \mid v_{i+3} = v_i \text{ for all } i \}$$

Prove that $U$ is a subspace of $F^\infty$.

Challenge Problem 6. Say that a sequence $v = (v_1, v_2, v_3, \ldots) \in F^\infty$ is periodic if there exists some positive number $k \in \mathbb{N}$ such that $v_{i+k} = v_i$ for all $i$. Let $W$ be the set of all periodic sequences:

$$W = \{v \in F^\infty \mid v \text{ is periodic} \}$$

Is $W$ a subspace of $F^\infty$? Prove or disprove.