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Geometric Literacy 3/8/10

Characteristic classes of surface bundles

1. surface bundles, moduli space
2. MMM classes
3. branched covers, signature
4. geometric classes, obstructions to flatness
5. overview of Madsen–Weiss

Surface bundle = fiber bundle $\begin{array}{c}\Sigma_g \to E\\ \downarrow\\ B\end{array}$ with fiber a surface Σ_g, $g \geq 2$.

Vector bundles have structure group $GL_n\mathbb{R}$.
Surface bundles have structure group $\mathrm{Diff}^+\Sigma_g$.

Fundamental fact: $\mathrm{Diff}_0\Sigma_g$ is contractible.

Why important? This means that
$$\mathrm{Diff}^+\Sigma_g \longrightarrow \underbrace{\mathrm{Diff}^+\Sigma_g/\mathrm{Diff}_0\Sigma_g}_{\parallel}$$
is a homotopy equivalence. $\pi_0\mathrm{Diff}^+\Sigma_g =: \mathrm{Mod}_g$

Compare: the fact that $O(n) \hookrightarrow GL_n\mathbb{R}$ is a homotopy equivalence says that vector-bundles are same as vector-bundles-with-metric.

Corresponding fact: Surface bundles are totally determined by their monodromy.

$$\left\{\begin{array}{c}\Sigma_g\text{-bundles}\\ \text{over } B\\ \text{up to isomorphism}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{representations}\\ \pi_1(B)\to\mathrm{Mod}_g\\ \text{up to conjugacy}\end{array}\right\}$$
(Both directions are nonobvious.)

What is monodromy?

Follow fiber around a loop:

first Stiefel-Whitney class $w_1: \pi_1(B) \to \mathbb{Z}/2\mathbb{Z}$
does orientation change?

monodromy $\pi_1(B) \to \mathrm{Mod}_g$
how does identification fiber $=\Sigma_g$ change?

Corollary: every surface bundle over a sphere (or any simply connected base) is trivial.

Moduli space – classifying space for Σ_g-bundles, analogue of Grassmanian.

Def. $\mathcal{M}_g$ is the space of all Riemann surfaces/ hyperbolic metrics/ conformal classes of metrics on a surface of genus g, up to isomorphism

$$\mathcal{M}_g := (\text{Metrics}/C^\infty(\Sigma_g,\mathbb{R}^+))/\mathrm{Diff}^+\Sigma_g$$

two metrics are conformal if $g_2 = \lambda g_1$ for some positive function λ, e.g. $g_{hyp} = \frac{1}{y^2} g_{euc}$

Fact. $\mathcal{M}_g$ is an orbifold $K(\mathrm{Mod}_g, 1)$

This means: $\widetilde{\mathcal{M}}_g := (\text{Metrics}/C^\infty(\Sigma_g,\mathbb{R}^+))/\mathrm{Diff}_0\Sigma_g$ is a contractible manifold,
$\mathrm{Mod}_g = \mathrm{Diff}^+/\mathrm{Diff}_0$ acts on $\widetilde{\mathcal{M}}_g$ with finite stabilizers
$\widetilde{\mathcal{M}}_g/\mathrm{Mod}_g = \mathcal{M}_g$, and so $\pi_1^{orb}(\mathcal{M}_g) = \mathrm{Mod}_g$

Why important? means $H^*(\mathcal{M}_g;\mathbb{Q}) = H^*(\mathrm{Mod}_g;\mathbb{Q}) = H^*(B\mathrm{Diff}^+\Sigma_g;\mathbb{Q})$ ← characteristic classes of surface bundles.

Proof sketch: hyperbolic geometry gives parametrization as finite-dimensional manifolds.
finite stabilizers ⟺ classical fact: any hyperbolic surface/ Riemann surface has finite isometry/ automorphism group.
(orbifold locus = exactly those surfaces with automorphisms)

$\widetilde{\mathcal{M}}_g$ contractible: Metrics is contractible (b/c convex: $tg_1 + (1-t)g_2$); $C^\infty(\Sigma_g,\mathbb{R}^+)$ is contractible (b/c convex), and $\mathrm{Diff}_0(\Sigma_g)$ is contractible (fundamental fact)

so $\widetilde{\mathcal{M}}_g = (\text{Metrics}/C^\infty(\Sigma_g,\mathbb{R}^+))/\mathrm{Diff}_0(\Sigma_g)$ is contractible.
(This is historical revisionism — this is actually how the fundamental fact is proved.)

Classifying map

Given $\begin{matrix}\Sigma_g \to E\\ \downarrow\\ M\end{matrix}$, what is the classifying map $M \longrightarrow \mathcal{M}_g$?

Put an arbitrary Riemannian metric on E.
Restriction to fiber over $m \in M$ gives a metric h_m on Σ_g.
So we send $m \in M$ to (the conformal class of) this metric in $\mathcal{M}_g$.

$$\begin{aligned} M &\longrightarrow \mathcal{M}_g \\ m &\longmapsto [h_m] \end{aligned}$$

Characteristic classes

How can we construct characteristic classes of $\begin{matrix}\Sigma_g \to E\\ \downarrow\\ M\end{matrix}$?

We have one natural vector bundle lying around: $\begin{matrix}\mathbb{R}^2 \to T^{vert}E\\ \downarrow\\ E\end{matrix}$, which has a characteristic class

$$e := eu(T^{vert}E) \in H^2(E)$$

Need something in $H^*(M)$, so we integrate along the fiber: $\int_{\Sigma_g} : H^k(E) \to H^{k-2}(M)$.

$\int_{\Sigma_g} e \in H^0(M)$ turns out to be the primary obstruction to the existence of a section $\begin{matrix}E\\ \uparrow\downarrow\\ M\end{matrix}$.

To get more classes, start with e^2, e^3, etc. (Integration along the fiber kills the relations between them.)

Definition (Morita-Mumford-Miller classes):
$e_i \in H^{2i}(\mathrm{Mod}_g)$ is the characteristic class of surface bundles defined on $\begin{matrix}\Sigma_g \to E\\ \downarrow\\ M\end{matrix}$ by $e_i\left(\begin{matrix}E\\ \downarrow\\ M\end{matrix}\right) = \int_{\Sigma_g} e^{k+1} \in H^{2i}(M)$.

Why should these be nontrivial?
Partial answer: consider $\begin{matrix}\Sigma_g \to E^4\\ \downarrow\\ M^2\end{matrix}$, and $e_1\left(\begin{matrix}E\\ \downarrow\\ M\end{matrix}\right) \in H^2(M)$. Then $\left\langle e_1\left(\begin{matrix}E\\ \downarrow\\ M\end{matrix}\right), [M]\right\rangle = 3 \cdot \mathrm{signature}(E^4)$.

Proof: Hirzebruch signature formula is truncated: $\mathrm{signature}(E^4) = \frac{1}{3}\int_E p_1(TE)$.

But note $TE = T^{vert}E \oplus \pi^*(TM)$

so $$\begin{aligned} p_1(TE) &= p_1(T^{vert}E) + p_1(\pi^*TM) \\ &\underset{\text{2-dim}}{=} eu(T^{vert}E)^2 + \cancel{\pi^*(\underbrace{p_1(TM)}_{H^4(M^2)})}^{\,0} = e^2 \end{aligned}$$

Thus $\mathrm{signature}(E) = \frac{1}{3}\int_E p_1(TE) = \frac{1}{3}\int_E e^2 = \frac{1}{3}\int_M \int_{\Sigma_g} e^2 = \frac{1}{3}\int_M e_1\left(\begin{matrix}E\\ \downarrow\\ M\end{matrix}\right)$, as claimed.

Theorem (Miller, Morita): $e_i \neq 0 \in H^{2i}(\mathrm{Mod}_g)$ for all $i \leq cg$. In fact, no polynomial relations in low degree: for any polynomial P in $e_1, e_2, \ldots$ of total degree $\ll g$, $P(e_1, e_2, \ldots) \neq 0$ (that is, $\mathbb{Q}[e_1, e_2, \ldots]$ injects into $H^*(\mathrm{Mod}_g)$ in low degrees).

This is proved by constructing surface bundles where these classes are nontrivial.
We will look at the simplest case: $e_1 \neq 0$. By the above, this is the same as the classical problem of finding a surface bundle over a surface with nonzero signature.

But didn't we see last week that any $\begin{matrix}\Sigma \to E^4\\ \downarrow\\ M^2\end{matrix}$ has $\mathrm{signature}(E) = 0$?
<u>No</u>; that was only if $\pi_1(M)$ acts trivially on $H^1(\Sigma)$.
So our examples will not have this property.

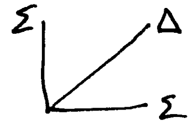
Recall: for $\begin{array}{c}\Sigma \to E^4 \\ \downarrow \\ M^2\end{array}$, $\langle e_1(E), [M]\rangle = \text{signature}(E)$.

Want to construct surface bundles over surfaces, i.e. families of surfaces parametrized by a surface.

Idea: over $p \in \Sigma$, take a branched cover of Σ branched only at p. $(z \mapsto z^n)$

Globally, start with trivial bundle, consider diagonal $\Delta \subset \Sigma \times \Sigma$.

$$\begin{tikzcd} \Sigma \arrow[r] & \Sigma \times \Sigma \arrow[d] \\ & \Sigma \end{tikzcd}$$



Let $\begin{array}{c} E \\ \downarrow (\Delta) \\ \Sigma\times\Sigma \end{array}$ be a branched cover of $\Sigma \times \Sigma$, branched over Δ.

The composition
$$\begin{tikzcd} S \arrow[r] \arrow[d, "(*)"] & E \arrow[d, dashed, "(\Delta)"] \\ \Sigma \arrow[r] & \Sigma\times\Sigma \arrow[d] \\ & \Sigma \end{tikzcd}$$
will be a surface bundle over Σ ~~whose~~ fiber S is itself a branched cover of Σ branched at one point.

Problem: You can't just take branched covers over whatever you want; in particular (fun exercise), no cyclic branched cover of Σ branched at one point; various solutions.

How can we know $\sigma(E) \neq 0$ or $e_1(E) \neq 0$?

1) Hirzebruch analyzed change in signature under branched covers, in this case expressible in terms of self-intersection number of Δ. Since $\sigma(\Sigma\times\Sigma) = 0$ and $\Delta \pitchfork \Delta = 2-2g$, $\sigma(E) \neq 0$.
(This is the approach that Morita, Miller use.)

2) $\mathcal{M}_g$ not just a orbifold, in fact a complex orbifold.
Just as surface bundles $\updownarrow$ maps $M \to \mathcal{M}_g$, holomorphic surface bundles $\updownarrow$ holomorphic maps $M \to \mathcal{M}_g$.

The construction above is nice and holomorphic, gives holomorphic map $i: \Sigma \to \mathcal{M}_g$ (non-constant by construction).

Theorem: $\mathcal{M}_g$ is in fact a Kähler orbifold, and the Kähler form ω represents $\frac{\pi^2}{6} e_1 \in H^2(\mathcal{M}_g)$.

Thus $\text{signature}(E) = \frac{1}{3}\int_\Sigma e_1 = \frac{2}{\pi^2}\int_\Sigma i^*\omega = \frac{2}{\pi^2}\int_{i(\Sigma)} \omega$.

But ~~Wirtinger~~ (name?) inequality says Kähler form on ~~complex~~ holomorphic submanifold always positive, so integrand is positive and $\text{Signature}(E) = \frac{2}{\pi}\int_{i(\Sigma)} \omega > 0$.

For $\Sigma \to E^4 \downarrow M^2$

We can think of $\langle e_1(E), [M] \rangle$ as a characteristic number for the fiber bundle.

For $\Sigma \to E^{12} \downarrow M^{10}$, we have $\langle e_5(E), [M] \rangle = |e_5|$.

The above shows that $|e_1|$ is geometric, meaning for another surface bundle

$\Sigma' \to E' \downarrow M'$, a homeomorphism of total spaces $E \simeq E'$

$\Rightarrow$ $|e_1(\substack{E\\ \downarrow\\ M})| = |e_1(\substack{E'\\ \downarrow\\ M'})| = 3\,\text{signature}(E)$.

Q: What about other characteristic numbers?

Theorem (Church-Farb-Thibault): the odd MMM classes are geometric:

For $\Sigma \to E^{2i+2} \downarrow M^{2i}$, the characteristic number $|e_i(\substack{E\\ \downarrow\\ M})| = \langle e_i(E), [M] \rangle$ only depends on E.

Q: Is every surface bundle flat?

A (Morita): No.

Thm: If a surface bundle $\Sigma \to E \downarrow M$ is flat, then $e_i(E) = 0$ for all $i \geq 3$.

Proof: uses Bott vanishing theorem.

Bott vanishing theorem: (obstruction for a bundle to be the normal bundle of a foliation).

If $\mathcal{F}$ is a codimension k foliation of M, with normal bundle $N\mathcal{F} \downarrow M$, then any combination of Pontryagin classes of $N\mathcal{F}$ vanishes in $H^i(M)$ for $i \geq 2k$.

If $\Sigma \to E \downarrow M$ were flat (admits a codim. 2 foliation $\mathcal{F}$ transverse to fibers), then $T^{vert}E$ would be the normal bundle. $\Rightarrow p_1(T^{vert}E)^2 = 0$.

But $p_1(T^{vert}E)^2 = e^4$, so $p_1^2 = 0$ implies $e_3 = \int_\Sigma e^4 = 0$, etc.

So bundles with nonzero e_i are not flat.

Q: Are all obstructions to flatness cohomological?

A: No.

Thm (Kotschick-Morita): every surface bundle $\Sigma \to E^4 \downarrow M^2$ can be "connect summed" with a trivial bundle $\Sigma \to \Sigma \times S \downarrow S$ to become flat.

Proof: recall flat $\Leftrightarrow$ monodromy lifts to representation $\pi_1(M) \to \text{Diff}^+\Sigma$.

It probably doesn't; but pick lifts of generators, the commutator relation maps to something in $\text{Diff}_0\Sigma$. But $\text{Diff}_0\Sigma$ is perfect (Thurston) so we may find a flat bundle over S to cancel this nonintegrability.

Homological stability:
for $g \geq 2i$, $H^i(\mathrm{Mod}_g; \mathbb{Q}) \simeq H^i(\mathrm{Mod}_{g+1}; \mathbb{Q})$.
So we can talk about the stable cohomology "$H^i(\mathrm{Mod}_\infty; \mathbb{Q})$"
MMM classes are stable
Mumford conjecture: $H^*(\mathrm{Mod}_\infty; \mathbb{Q}) = \mathbb{Q}[e_1, e_2, e_3, \dots]$.
Proved by Madsen-Weiss – all stable characteristic classes come from e_i.

Idea:
A surface bundle $\Sigma \to E \xrightarrow{\pi} M$ gives a splitting $TE = T^{vert}E \oplus \pi^* TM$.
We'll say that a "formal surface bundle" is just this data:
a smooth map $E \xrightarrow{\pi} M$, and a 2-dim. bundle ζ ~~so that~~ with an isomorphism $TE = \zeta \oplus \pi^* TM$ stably.
This DOESN'T have to be a bundle (e.g. π may not be a submersion).
It was already known that the only char. classes of formal surface bundles are $e_1, e_2, e_3, \dots$, so it suffices to prove
Theorem (Eliashberg, Galatius, Mishachev): Every formal surface bundle is cobordant to an actual surface bundle.

How are formal surface bundles classified? Let $\mathbb{C} \to L_N \to \mathbb{CP}^n$ and $\mathbb{C}^n \to L_n^\perp \to \mathbb{CP}^n$ be tautological bundles.
Consider a map $M \times \mathbb{R}^{2N+2} \xrightarrow{f} L_N^\perp$.
Assuming f is transverse to zero section, preimage of $\mathbb{CP}^N \subset L_N^\perp$ is codim. $2N$ submanifold of $M \times \mathbb{R}^{2N+2}$, call it $E := f^{-1}(\mathbb{CP}^N)$
projection $M \times \mathbb{R}^{2N+2} \to M$ gives $E \xrightarrow{\pi} M$, and let $\zeta := f^*(L_N)$.
If NE is normal bundle to E, note $NE = f^*(L_N^\perp)$.
By definition, $T(M \times \mathbb{R}^{2N+2})|_E = TE \oplus NE$. Now add ζ to both sides
$$T(M \times \mathbb{R}^{2N+2}) = \pi^* TM \oplus \mathbb{R}^{2N+2}$$
$$\Rightarrow \pi^* TM \oplus \zeta \oplus \mathbb{R}^{2N+2} = TE \oplus f^*(L_N^\perp) \oplus f^*(L_N) \qquad L_N^\perp \oplus L_N = \mathbb{C}^{n+1}$$
$$= TE \oplus \mathbb{R}^{2N+2}, \text{ a stable isomorphism.}$$
One-point compactify, $M \times \mathbb{R}^{2N+2} \cup \{\infty\} = \Sigma^{2N+2} M \longrightarrow Th(L_N^\perp) = L_n^\perp \cup \{\infty\}$.
By Σ-Ω adjunction, same as $M \longrightarrow \Omega^{2N+2} Th(L_N^\perp)$.
Formal surface bundles classified by maps to $\varinjlim \Omega^{2N+2} Th(L_N^\perp)$

Theorem (MW): $\mathbb{Z} \times B\mathrm{Mod}_g^+ \longrightarrow \varinjlim \Omega^{2N+2} Th(L_N^\perp)$ is a homotopy equivalence.