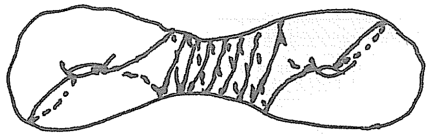
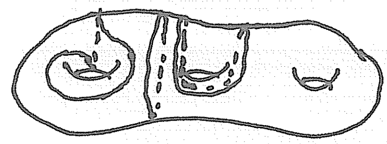
Thanks. 3 parts.

Question: How many different curves can you draw on a surface?

curve = embedded S^1, up to isotopy

"different"?





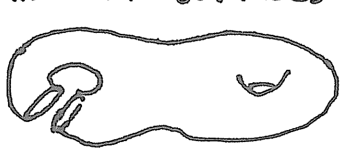
$\gamma \sim \delta \Leftrightarrow \exists \varphi \in \mathrm{Mod}(\Sigma),\ \varphi(\gamma) = \delta$

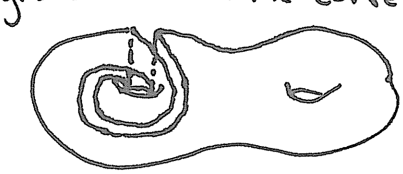
$\underline{S^2}$: Jordan curve theorem

$D^2 \cup_\gamma D^2 = S^2 = D^2 \cup_\delta D^2$

build $\varphi \in \mathrm{Homeo}(S^2)$, $\varphi(\gamma) = \delta$

$\underline{\Sigma}$: Classification of surfaces (ignore contractible curves)





Mod	nonsep.	sep.
	none	genus k cut off

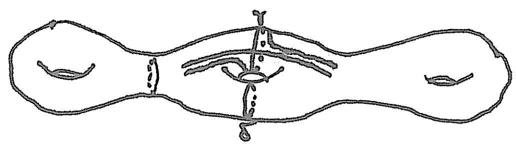
Restrict "same" — put condition not on curves, but on φ

Consider action of $\mathrm{Mod}(\Sigma)$ on $H_1(\Sigma)=H$

alg. intersection $(\cdot,\cdot): H_1(\Sigma)\times H_1(\Sigma)\to\mathbb{Z}$ symplectic

$$1\to\mathcal{I}(\Sigma)\to\mathrm{Mod}(\Sigma)\xrightarrow{A}Sp_{2g}\mathbb{Z}\to 1$$

Torelli group



How does a Dehn twist act on H?

$T_\gamma(h)=h+(h,[\gamma])[\gamma]$

γ separating $\Rightarrow [\gamma]=0 = T_\gamma\in\mathcal{I}(\Sigma)$ "separating twist"

$[\gamma]=[\delta]\Rightarrow T_\gamma, T_\delta$ act same on $H\Rightarrow T_\gamma T_\delta^{-1}\in\mathcal{I}(\Sigma)$ "bounding pair"

[Thm (Birman, Powell): $\mathcal{I}(\Sigma)$ gen. by sep. twists and bounding pairs

Define: $\gamma\sim_{\mathcal{I}}\delta \Leftrightarrow \exists\varphi\in\mathcal{I}(\Sigma),\ \varphi(\gamma)=\delta$

Obvious necessary conditions for $\gamma\sim_{\mathcal{I}}\delta$:

$\gamma\sim_{\mathcal{I}}\delta\Rightarrow[\gamma]=[\delta]$

for separating γ, let $V_\gamma=H_1(\ )\le H_1(\ )$

$f_*V_\gamma=V_{f(\gamma)}$, so $\gamma\sim_{\mathcal{I}}\delta\Rightarrow V_\gamma=V_\delta$

Theorem (Johnson): these conditions are sufficient.

$\mathcal{I}(\Sigma)$	$[\gamma]\in H$	$V_\gamma\le H$

Why is $A: \mathrm{Mod}(\Sigma) \to Sp_{2g}\mathbb{Z}$ onto?
$x \longmapsto x + (x,y)y$ is a transvection; they generate $Sp_{2g}\mathbb{Z}$
Just need to find enough Dehn twists.

Proof of Johnson's theorem for separating twists

Given γ, δ so that $V_\gamma = V_\delta$, need to find $\varphi \in \mathcal{I}(\Sigma)$ taking γ to δ.
We know $\exists f \in \mathrm{Mod}(\Sigma)$ taking γ to δ.
It suffices to find $g \in \mathrm{Stab}(\gamma)$ s.t. $A(g) = A(f)$.
Why? Then $\varphi = fg^{-1}$ is the desired element of $\mathcal{I}(\Sigma)$. (Check.)

To find g:
$f(\gamma) = \delta \Rightarrow f_* V_\gamma = V_\delta = V_\gamma$, also $f_* V^\perp = V^\perp$ so $A(f)$ looks like $\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$
show that $A(\mathrm{Stab}(\gamma)) = \left(\begin{array}{c|c} Sp_{2k}\mathbb{Z} & 0 \\ \hline 0 & Sp_{2\ell}\mathbb{Z} \end{array}\right)$ by finding enough Dehn twists on each side of γ. ∎

$K(\Sigma)$ = group gen. by separating twists
$\gamma \sim_K \delta$ requires $\gamma \sim_{\mathcal{I}} \delta$ because $K(\Sigma) < \mathcal{I}(\Sigma)$
nonseparating case:
realize γ, δ by elements of $\pi = \pi_1(\Sigma)$
$[\gamma] = [\delta] = X \Rightarrow \gamma\delta^{-1} \in [\pi, \pi]$

$$\text{consider } [\gamma\delta^{-1}] \in [\pi,\pi]/[\pi,[\pi,\pi]] \approx \Lambda^2 H$$
$$[\alpha,\beta] \longmapsto [\alpha] \wedge [\beta]$$

Theorem (Church): $\gamma \sim_K \delta \iff [\gamma\delta^{-1}]$ lies in $X \wedge X^\perp \leq \Lambda^2 H$

Note: $\varphi T_\gamma \varphi^{-1} = T_{\varphi(\gamma)}$

so solves conjugacy problem for Dehn twists
in $\mathrm{Mod}(\Sigma)$, in $\mathcal{I}(\Sigma)$, in $K(\Sigma)$