Joint work with Mladen Bestvina and Juan Souto. ①
Thanks.
Structure of talk: background + results, geometric perspective, theorem.

## Realization Problem

$$\mathrm{Mod}(\Sigma_g) = \mathrm{Diff}(\Sigma_g)/\text{isotopy} = \mathrm{Diff}(\Sigma_g)/\mathrm{Diff}_0(\Sigma_g)$$

"mapping class group"

$$\begin{tikzcd} & \mathrm{Diff}(\Sigma_g) \arrow[d] \\ G \arrow[ur, dashed] \arrow[r, hook] & \mathrm{Mod}(\Sigma_g) \end{tikzcd}$$

Which subgroups $G$ lift?

- $G$ finite (Nielsen realization problem), any $G$ lifts (Kerckhoff)
- $G \approx \mathbb{Z}^n$, any $G$ lifts (Birman-Lubotzky-McCarthy, Ivanov)

Q: Does every subgroup lift? That is, does $\mathrm{Mod}(\Sigma_g)$?
A: No. (Morita)

Morita showed that for $g \geq 18$, $\mathrm{Mod}(\Sigma_g)$ doesn't lift.
Improved to $g \geq 3$ by Franks-Handel.
Markovic(-Saric): for $g \geq 2$, $\mathrm{Mod}(\Sigma_g)$ doesn't even lift to $\mathrm{Homeo}(\Sigma_g)$.

Open question: is there any $\pi_1(\Sigma_h) \hookrightarrow \mathrm{Mod}(\Sigma_g)$ that doesn't lift?
(see later, has natural geometric interpretation)

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Fix $* \in \Sigma_g$. $\quad \mathrm{Mod}(\Sigma_g, *) = \mathrm{Diff}(\Sigma_g, *)/\mathrm{Diff}_0(\Sigma_g, *)$

Birman, $g \geq 2$: $\quad 1 \to \underbrace{\pi_1(\Sigma_g) \xrightarrow{PP} \mathrm{Mod}(\Sigma_g, *)}_{\text{"point-pushing subgroup"}} \longrightarrow \mathrm{Mod}(\Sigma_g) \to 1$

**Theorem A:** $PP: \pi_1(\Sigma_g) \hookrightarrow \mathrm{Mod}(\Sigma, *)$ does not lift to $\mathrm{Diff}(\Sigma_g, *)$, $g \geq 2$.

**Theorem B:** $\exists\ \pi_1(\Sigma_h) \times \mathbb{Z}/3\mathbb{Z} \hookrightarrow \mathrm{Mod}(\Sigma_g)$ that does not lift to $\mathrm{Diff}(\Sigma_g)$, $g \geq 8$, $g \geq 6$.

Corollary: Morita's theorem.
Remark: this $\pi_1(\Sigma_h) \hookrightarrow \mathrm{Mod}(\Sigma_g)$ is the monodromy of an Atiyah-Kodaira manifold ($\Sigma_g$-bundle over $\Sigma_h$ w/ nonzero signature)

# Surface bundles (geometric perspective)

$$\begin{array}{ccc} \Sigma_g & \to & E \\ & & \downarrow \\ & & B \end{array}$$

standing assumptions:
- $g \geq 2$
- all bundles orientable, all diffeos orientation-preserving, etc.

## One construction:

Given $\rho: \pi_1(B) \to \mathrm{Diff}(\Sigma_g)$,  $\pi_1(B) \circlearrowright \underset{\text{d.t.}}{\widetilde{B}} \times \underset{\rho}{\Sigma_g}$  covering action

$$\begin{array}{ccl} \Sigma_g & \to & E_\rho = (\widetilde{B} \times \Sigma_g)/\pi_1(B) \\ & & \downarrow \\ & & B \end{array}$$

$E_\rho$ is the $\Sigma_g$-bundle <u>associated to $\rho$</u>.

We say a bundle $E$ is <u>flat</u> if $E \cong E_\rho$ for some $\rho$.

($\Leftrightarrow \exists$ a flat connection on $E$)

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## Monodromy

For any $\begin{array}{ccc}\Sigma_g & \to & E \\ & & \downarrow \\ & & B\end{array}$ we have $r: \pi_1(B) \to \mathrm{Mod}(\Sigma_g)$ defined up to conjugacy.

$r$ is the <u>monodromy</u> of the bundle $E$.

Note: For a flat bundle $E_\rho$, monodromy is the composition 
$$\begin{array}{ccc} & \overset{\rho}{\nearrow} & \mathrm{Diff}(\Sigma_g) \\ \pi_1(B) & & \downarrow \\ & \underset{r}{\to} & \mathrm{Mod}(\Sigma_g) \end{array}$$

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Amazing fact: $\mathrm{Diff}_0(\Sigma_g)$ is contractible.

Amazing fact, take II: Surface bundles are determined by their monodromy.

$\left\{\begin{array}{c}\Sigma_g\text{-bundles} \\ \text{over } B \text{ up to} \\ \text{isomorphism}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{representations} \\ r: \pi_1(B) \to \mathrm{Mod}(\Sigma_g) \\ \text{up to conjugacy}\end{array}\right\}$

flat bundles ($\exists$ flat connection) $\longleftrightarrow$ reps. $r$ that lift to $\rho$

- if $r_E = r_{E'}$, $E \cong E'$
- for any $r: \pi_1(B) \to \mathrm{Mod}(\Sigma_g)$, $\exists \begin{array}{ccc}\Sigma_g & \to & E \\ & & \downarrow \\ & & B\end{array}$ w/ monodromy $r$. (if $r$ does not lift to $\rho$, not clear how to build $E$)

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$\left\{\begin{array}{c}\Sigma_g\text{-bundles} \\ \text{over } B \\ \text{with distinguished section} \\ \text{up to isomorphism}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{representations} \\ r: \pi_1(B) \to \mathrm{Mod}(\Sigma_g, *) \\ \text{up to conjugacy}\end{array}\right\}$

flat bundles-with-section ($\exists$ flat connection s.t. section is parallel) $\longleftrightarrow$ reps. that lift

Morita's theorem: there are non-flat surface bundles.

Open question (restated): is there a 4-manifold $\begin{array}{c}\Sigma_g \to M^4\\ \downarrow\\ \Sigma_h\end{array}$ that admits no flat connection?

## Proof of Theorem A

Necessary background: circle bundles

$$\mathrm{Homeo}^+(S^1) \simeq GL_2^+\mathbb{R} \simeq \mathbb{C}^*$$

$$\left\{\begin{array}{c}\text{circle bundles}\\ \text{over } B\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{2-dimensional}\\ \text{real vector bundles}\\ \text{over } B\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{complex line bundles}\\ \text{over } B\end{array}\right\}$$

same characteristic class: Euler class

$\begin{array}{c}S^1 \to E\\ \downarrow\\ B\end{array}$ or $\begin{array}{c}\mathbb{R}^2 \to E\\ \downarrow\\ B\end{array}$ or $\begin{array}{c}\mathbb{C} \to E\\ \downarrow\\ B\end{array}$ $\quad e(E) \in H^2(B)$

if $B = \Sigma$, pair with $[\Sigma] \in H_2(B)$ to get Euler number $eu(E) \in \mathbb{Z}$

Given e.g. $\rho: \pi_1(B) \to \mathrm{Homeo}^+(S^1)$, get associated bundles $\begin{array}{c}S^1 \to E_\rho\\ \downarrow\\ B\end{array}$

Such bundles are called flat (if $E \simeq E_\rho$). (no monodromy condition)

- Milnor-Wood inequality
  If $\begin{array}{c}S^1 \to E\\ \downarrow\\ \Sigma_g\end{array}$ is flat, then $|eu(E)| \leq 2g-2$.
- Milnor's inequality
  If $\begin{array}{c}\mathbb{R}^2 \to E\\ \downarrow\\ \Sigma_g\end{array}$ is flat, then $|eu(E)| \leq g-1$.
- Fact (Deligne-Sullivan)
  If $\begin{array}{c}\mathbb{C} \to E\\ \downarrow\\ \Sigma_g\end{array}$ is flat, then $eu(E) = 0$ + $E$ is trivial.

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$UT\Sigma_g$ is flat:

$\pi_1(\Sigma_g) \curvearrowright \mathbb{H}^2$, $G \curvearrowright \partial\mathbb{H}^2$

$\rho: \pi_1(\Sigma_g) \to \mathrm{Homeo}^+(S^1)$

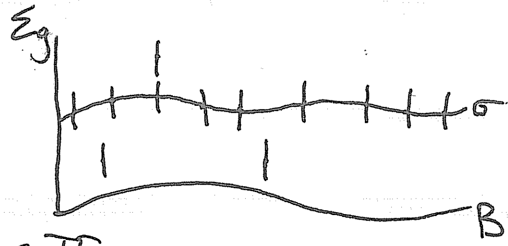
$UT\Sigma_g \simeq E_\rho$

$T\Sigma_g$ is not flat

$eu(UT\Sigma_g) = 2-2g = eu(T\Sigma_g)$

# Vertical Euler class:

Given any $\begin{array}{c}\Sigma_g \to E \circlearrowleft \sigma \\ \downarrow \\ B\end{array}$,




vectors in TE tangent to fibers give $T^{vert}E \subset TE$.

If we have a section $\sigma : B \to E$, can restrict $T^{vert}E$ to $\sigma$, giving $\begin{array}{c}\mathbb{R}^2 \to T^{vert}\sigma \\ \downarrow \\ B\end{array}$

Could take $e(T^{vert}E) \in H^2(E)$, but want something in $H^*(B)$

$e_{vert}(E,\sigma) = e(T^{vert}\sigma) \in H^2(B)$

If $B = \Sigma$, pair with $[\Sigma]$ to get $eu_{vert}(E,\sigma) \in \mathbb{Z}$

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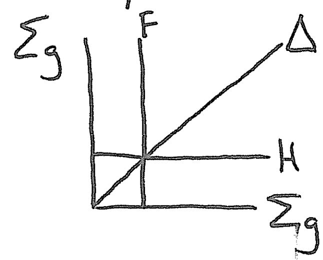
**Theorem A:** $PP: \pi_1(\Sigma_g) \to \mathrm{Mod}(\Sigma_g, *)$ does not lift to $\mathrm{Diff}(\Sigma_g, *)$.

Q: What bundle-with-section has monodromy PP?

A: $(\Sigma_g \times \Sigma_g, \Delta)$




$T^{vert}F \cong TF$

$T^{vert}H = \mathbb{R}^2 \times H$

$T^{vert}\Delta \cong T\Delta$

Pf:

Step 1: $T^{vert}\Delta \cong T\Delta$

Step 2: If $(E,\sigma)$ is flat, then $T^{vert}\sigma$ is flat.

Step 3: Contradiction:

If PP lifts, $(\Sigma_g \times \Sigma_g, \Delta)$ is flat. So $T^{vert}\Delta$ is flat (Step 2).

Milnor's ineq. $\Rightarrow |eu(T^{vert}\Delta)| \leq g-1$. But $eu(T^{vert}\Delta) = eu(T\Delta) = 2-2g$.

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## Milnor's inequality II:

Given $\begin{array}{c}\Sigma_g \to E \circlearrowleft \sigma \\ \downarrow \\ \Sigma_h\end{array}$, if $(E,\sigma)$ is flat, then $|eu_{vert}(E,\sigma)| \leq h-1$.