Joint work with Mladen Bestvina and Juan Souto.

Thanks.

Structure of talk: background + results, geometric perspective, theorem.

Realization Problem

\[ \text{Mod}(\Sigma_g) = \frac{\text{Diff}(\Sigma_g)}{\text{isotopy}} = \frac{\text{Diff}(\Sigma_g)}{\text{Diff}_0(\Sigma_g)} \]

*mapping class group*

Which subgroups \( G \) lift?

- \( G \) finite (Nielsen realization problem), any \( G \) lifts (Kerckhoff)
- \( G \cong \mathbb{Z}^n \), any \( G \) lifts
  (Birman-Lubotzky-McCarthy, Ivanov)

Q: Does every subgroup lift? That is, does \( \text{Mod}(\Sigma_g) \)?

A: No. (Morita)

Morita showed that for \( g \geq 18 \), \( \text{Mod}(\Sigma_g) \) doesn't lift.

Improved to \( g \geq 3 \) by Franks-Handel.

Markovic-Saric: for \( g \geq 2 \), \( \text{Mod}(\Sigma_g) \) doesn't even lift to \( \text{Homeo}(\Sigma_g) \).

Open question: is there any \( \pi_1(\Sigma_h) \to \text{Mod}(\Sigma_g) \) that doesn't lift?
  (See later, has natural geometric interpretation)

Fix \(* \in \Sigma_g \). \( \text{Mod}(\Sigma_g,* \) = \( \frac{\text{Diff}(\Sigma_g,* \)}{\text{Diff}(\Sigma_g,*)} \)

Birman, \( g \geq 2 \): \( 1 \to \pi_1(\Sigma_g) \xrightarrow{pp} \text{Mod}(\Sigma_g,*) \to \text{Mod}(\Sigma_g) \to 1 \)

*point-pushing subgroup*

Theorem A: \( \text{PP}: \pi_1(\Sigma_g) \to \text{Mod}(\Sigma_g,*) \) does not lift to \( \text{Diff}(\Sigma_g,*) \), \( g \geq 2 \).

Theorem B: \( \exists \pi_1(\Sigma_h) \times \mathbb{Z}/3\mathbb{Z} \to \text{Mod}(\Sigma_g) \) that does not lift to \( \text{Diff}(\Sigma_g) \), \( g \geq 8, g \geq 6 \).

Corollary: Morita's theorem.

Remark: this \( \pi_1(\Sigma_h) \to \text{Mod}(\Sigma_g) \) is the monodromy of
an Atiyah-Kodaira manifold (\( \Sigma_g \)-bundle over \( \Sigma_h \) w/ nonzero signature).
Surface bundles (geometric perspective)

\[ \Sigma_g \rightarrow E \quad \text{over } B \]

Standing assumptions: \( g \geq 2 \)
- all bundles orientable,
- all diffeos orientation-preserving, etc.

One construction:
Given \( p: \pi_1(B) \rightarrow \text{Diff}_0(\Sigma_g) \),
\( \pi_1(B) \curvearrowright \tilde{B} \times \Sigma_g \) covering action
d.t. \( p \)

\[ \Sigma_g \rightarrow E_p = (\tilde{B} \times \Sigma_g) / \pi_1(B) \]
\[ \downarrow \]
\[ B \]

\( E_p \) is the \( \Sigma_g \)-bundle associated to \( p \).

We say a bundle \( E \) is flat if \( E \cong E_p \) for some \( p \).

\[ \iff \exists \text{ a flat connection on } E \]

Monodromy

For any \( \Sigma_g \rightarrow B \) we have \( r: \pi_1(B) \rightarrow \text{Mod}(\Sigma_g) \) defined up to conjugacy.

\( r \) is the monodromy of the bundle \( E \).

Note: For a flat bundle \( E_p \), monodromy is the composition \( \pi_1(B) \rightarrow \text{Mod}(\Sigma_g) \).

Amazing fact: \( \text{Diff}_0(\Sigma_g) \) is contractible.

Amazing fact, take II: Surface bundles are determined by their monodromy.

\[ \{ \text{\( \Sigma_g \)-bundles over } B \text{ up to isomorphism} \} \leftrightarrow \{ \text{representations } r: \pi_1(B) \rightarrow \text{Mod}(\Sigma_g) \} \text{ up to conjugacy} \]

\[ \{ \text{flat bundles} \} \leftrightarrow \{ \text{representations } r: \pi_1(B) \rightarrow \text{Mod}(\Sigma_g) \text{ that lift to } p \} \]

\[ \{ \text{\( \Sigma_g \)-bundles over } B \text{ with distinguished section up to isomorphism} \} \leftrightarrow \{ \text{representations } r: \pi_1(B) \rightarrow \text{Mod}(\Sigma_g) \text{ that lift} \} \]

- if \( r_E = r_{E'} \), \( E \cong E' \)
- for any \( r: \pi_1(B) \rightarrow \text{Mod}(\Sigma_g) \),
  \( \exists \Sigma_g \rightarrow B \) w/ monodromy \( r \).
  (If \( r \) does not lift to \( p \), not clear how to build \( E \))
Morita's theorem: there are non-flat surface bundles.

Open question (restated): is there a 4-manifold \( \Sigma_g \to M^n \)
that admits no flat connection?

Proof of Theorem A

Necessary background: circle bundles

\[
\text{Homeo}^+(S^1) \cong GL^+_2 \mathbb{R} \cong C^*
\]

\[
\begin{cases}
\text{circle bundles} & \leftrightarrow \text{2-dimensional real vector bundles} & \leftrightarrow \text{complex line bundles} \\
\text{over } B & \text{over } B & \text{over } B
\end{cases}
\]

same characteristic class: Euler class

\[
\begin{align*}
S^1 & \to E & B \\
\mathbb{R} & \to E & B \\
\mathbb{C} & \to E & B
\end{align*}
\]

if \( B = \Sigma \), pair with \([\Sigma] \in H_2(B)\)
to get Euler number \( e(E) \in \mathbb{Z} \)

Given e.g. \( p: \pi_1(B) \to \text{Homeo}^+(S^1) \), get associated bundles \( E_p \)
Such bundles are called \( \text{flat} \) (if \( E = E_p \)). (no monodromy condition)

- Milnor-Wood inequality

  If \( S^1 \to E \)
  is flat, then \( |\text{ev}(E)| \leq 2g - 2 \).

- Milnor's inequality

  If \( \mathbb{R} \to E \)
  is flat, then \( |\text{ev}(E)| \leq g - 1 \).

- Fact (Deligne-Sullivan)

  If \( \mathbb{C} \to E \)
  is flat, then \( \text{ev}(E) = 0 \) + \( E \) is trivial.

\( UT\Sigma_g \) is flat:

\[
\begin{align*}
\pi_i(\Sigma_g) & \in H^2, \ 	ext{and} H^2 \\
p: \pi_1(\Sigma_g) & \to \text{Homeo}^+(S^1) \\
UT\Sigma_g & \cong E_p & \text{ev}(UT\Sigma_g) = 2 - 2g = \text{ev}(T\Sigma_g)
\end{align*}
\]

\( T\Sigma_g \) is not flat
Vertical Euler class:

Given any $\Sigma \to E \overset{\pi}{\to} B$,
 vectors in $TE$ tangent to fibers give $T^\text{vert}E \subset TE$.

If we have a section $\sigma : B \to E$, can restrict $T^\text{vert}E$ to $\sigma$,
 giving $\mathbb{R}^2 \to T^\text{vert}\sigma \downarrow B$

$e_{\text{vert}}(E, \sigma) = 1 \in H^2(B)$

If $B = \Sigma$, pair with $[\Sigma]$ to get $e_{\text{vert}}(E, \sigma) \in \mathbb{Z}$

Theorem A: PP: $\pi_1(\Sigma) \to \text{Mod}(\Sigma, \#)$ does not lift to $\text{Diff}(\Sigma, \#)$.

Q: What bundle-with-section has monodromy PP?

A: $(\Sigma \times \Sigma, \Delta)$

PF:

Step 1: $T^\text{vert}\Delta \cong T\Delta$

Step 2: If $(E, \sigma)$ is flat, then $T^\text{vert}\sigma$ is flat.

Step 3: Contradiction:

If PP lifts, $(\Sigma \times \Sigma, \Delta)$ is flat. So $T^\text{vert}\Delta$ is flat (Step 2).

Milnor's ineq. $|\text{ev}(T^\text{vert}\Delta)| \leq g - 1$. But $\text{ev}(T^\text{vert}\Delta) = \text{ev}(T\Delta) = 2 - 2g$.

Milnor's inequality II:

Given $\Sigma \to E \overset{\pi}{\to} B$, if $(E, \sigma)$ is flat, then $|\text{ev}_{\text{vert}}(E, \sigma)| \leq h - 1$. 