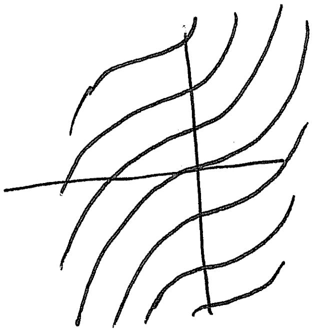
Geometric Literacy 2/24/10 — Tom Church

Flat connections: Lie theory, gauge theory, knot theory

Thurston: The derivative of a real-valued function f on a domain D is the Lagrangian section of T^*D giving the connection form of the unique flat connection on the trivial $\mathbb{R}$-bundle $D \times \mathbb{R}$ so that the graph of f is parallel.

Consider $f(x)=x^3$, look at foliation of plane by $f(x)+c$:

Slope at (x,y) is $f'(x)$



can reconstruct f from the level curves of foliation.

Goal: H, G simply connected Lie groups.

$F: H \to G$ induces $f: \mathfrak{h} \to \mathfrak{g}$

Theorem: i) F is uniquely determined by f, and

ii) every Lie algebra homomorphism f gives a Lie group homomorphism F

Beautiful insight: for M simply connected,

$$\mathrm{Maps}(M,G) = \{\omega \in \Omega^1(M;\mathfrak{g}) \mid d\omega + \tfrac{1}{2}[\omega,\omega]=0\}$$

$$= \{\text{flat connections on trivial bundle } M\times G \to M\}$$

Just as $f:\mathbb{R}\to\mathbb{R}$ corresponds to $df = f^*dx$, $F: M \to G$ will correspond to $F^*\omega_0$. But we need an integrability condition.

How can we see that (MC) $d\omega + \frac{1}{2}[\omega,\omega]=0$ is the right condition? Because both are equal to:

Start from the beginning:

Recall $\omega_0 \in \Omega^1(G,\mathfrak{g})$ "tautological form"

(MC) $d\omega_0 + \frac{1}{2}[\omega_0,\omega_0]=0$

Proof: To check at a point, can choose any vector fields. We take L_x, L_y for $x,y\in\mathfrak{g}$.

Recall $d\eta(V,W) = \frac{1}{2}\partial_W \eta(V) - \partial_V \eta(W) - \eta([V,W])$.

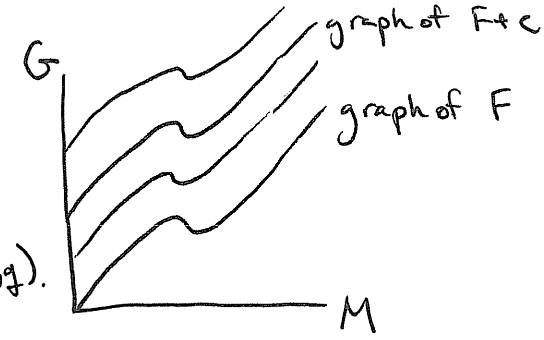
So $d\omega_0(L_x,L_y) = \frac{1}{2}\partial_{L_y}\omega_0(L_x) - \partial_{L_x}\omega_0(L_y) - \omega_0([L_x,L_y])$

But $\omega_0(L_x) = x$ ← constant function and $[L_x,L_y] = L_{[x,y]}$

so $= \frac{1}{2}\quad 0 - 0 - \omega_0(L_{[x,y]})$

$= -\frac{1}{2}[x,y] = -\frac{1}{2}[\omega_0,\omega_0](L_x,L_y)$. ■

How should $\mathrm{Map}(M,G) \xrightarrow{?} \{\text{flat connections}\}$



What is explicit connection form on $M \times G$?

Take simplest example, $\mathrm{id}: G \to G$ with form $\omega_0 \in \Omega^1(G; \mathfrak{g})$.

"parallel" w.r.t. this connection should mean "traveling same way in fiber as in base"

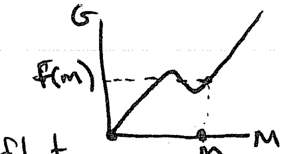
connection form is $(\omega_0)_{\text{fiber}} - (\omega_0)_{\text{base}}$.

In general, given $\omega \in \Omega^1(M; \mathfrak{g})$, define connection $\bar{\omega}$ on $M \times G$ by $\bar{\omega} = (\omega_0)_{\text{fiber}} - (\omega)_{\text{base}}$

Note (MC) for $\omega \Leftrightarrow$ (MC) for $\bar{\omega} \Leftrightarrow$ connection $\bar{\omega}$ is flat

will prove shortly

Now parallel transport from $* \in M$ to $m \in M$ sends e in fiber over $*$ to some point in fiber over m



call it $F(m) \in G$ (well-defined locally because connection is flat and globally because M is simply connected)

We have reconstructed $F: M \to G$ from parallel transport of connection $\bar{\omega}$ defined from $\omega = F^* \omega_0$.

$$\begin{tikzcd} F \arrow[rr] & & \omega = F^*\omega_0 \arrow[dl] \\ & \bar{\omega} \arrow[ul] & \end{tikzcd}$$

Recall flat means horizontal distribution is integrable.

Proof that flat $\Leftrightarrow \Omega = 0$:

Frobenius integrability theorem: a distribution is integrable $\Leftrightarrow$ given any two vector fields tangent to distribution, their bracket is tangent to distribution

So we need to prove $\Omega = 0 \Leftrightarrow$ bracket of horizontal vector fields is horizontal.

Let X, Y be any horizontal vector fields, meaning $\omega(X) = \omega(Y) = 0$.

By definition, $d\omega + \frac{1}{2}[\omega,\omega] = \Omega$, so $d\omega(X,Y) + \frac{1}{2}[\omega,\omega](X,Y) = \Omega(X,Y)$

$\Rightarrow d\omega(X,Y) = \Omega(X,Y)$

But in general, $d\omega(X,Y) = \frac{1}{2}\left(\partial_X \omega(Y) - \partial_Y \omega(X) - \omega([X,Y])\right)$

$= \cancel{}^{0} - \cancel{}^{0} - \frac{1}{2}\omega([X,Y])$.

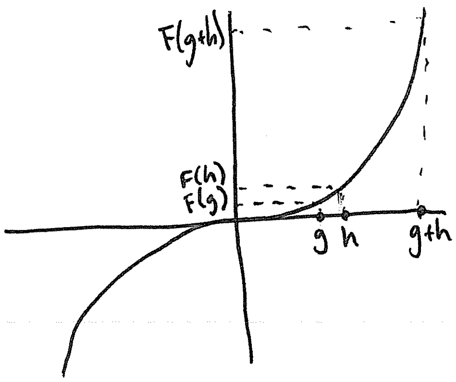
Thus $\Omega(X,Y) = 0 \Leftrightarrow \omega([X,Y]) = 0 \Leftrightarrow [X,Y]$ horizontal, as desired. ∎

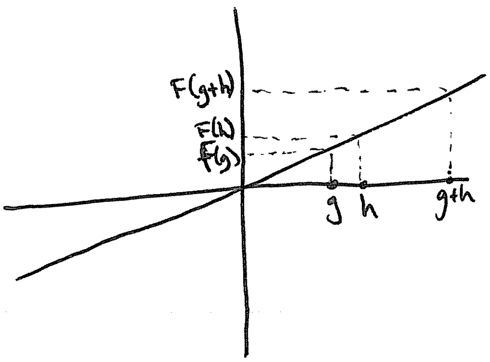
Consider $M = H$ a Lie group.

$$\mathrm{Maps}(H,G) \longleftrightarrow \{\omega \in \Omega^1(H;\mathfrak{g}) \mid d\omega + \tfrac{1}{2}[\omega,\omega] = 0\}$$

What about Lie group homomorphisms? $\cup$

$$\begin{array}{ccc} \cup & & \cup \\ \mathrm{Hom}(H,G) & \longleftrightarrow & ? \end{array}$$





Increase from e to g should be same as from h to gh $\Rightarrow$ ω should be left invariant

$$\mathrm{Hom}(H,G) \longleftrightarrow \left\{\omega \in \Omega^1(H,\mathfrak{g}) \;\middle|\; \begin{array}{l}\omega \text{ left-invariant and} \\ d\omega + \tfrac{1}{2}[\omega,\omega] = 0\end{array}\right\}$$

but left-invariant $\Rightarrow$ ω determined by value on $T_eH = \mathfrak{h}$
and (MC) $\Rightarrow$ map $\omega|_{T_eH} : \mathfrak{h} \to \mathfrak{g}$ preserves bracket

so $\mathrm{Hom}(H,G) \longleftrightarrow \mathrm{Hom}(\mathfrak{h},\mathfrak{g})$

$$F \xmapsto[\text{derivative at identity}]{} f$$

$$F \longleftarrow f$$

parallel transport

connection $(\omega_0^G)_{\text{fiber}} - (f \circ \omega_0^H)_{\text{base}}$ on $H \times G$

Chern-Simons form

Let P(matrix) be the invariant polynomial of degree 2 given by

$$\det(I+tA) = 1 + (\operatorname{tr} A)t + P(A)t^2 + \dots$$

We saw last time that on any principal G-bundle with connection ω,

$P(\Omega) \in \Omega^4(P;\mathbb{R})$ descends to $\widetilde{P(\Omega)} \in \Omega^4(M;\mathbb{R})$ with $[\widetilde{P(\Omega)}] = p_1$ in $H^4(M;\mathbb{R})$

But $P(\Omega)$ itself is exact on P

because $P(\Omega)$ represents p_1 of pullback bundle $G \to P\times G \to P$. But this is a trivial bundle, so $p_1 = 0$ in $H^4(P;\mathbb{R})$.

Chern-Simons found canonical $CS(\omega) \in \Omega^3(P;\mathbb{R})$ so that $d\, CS(\omega) = P(\Omega)$, namely $CS(\omega) = \operatorname{tr}(d\omega\wedge\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega)$

Chern-Simons theory

Let M^3 be a closed 3-manifold and $G = SU(2)$.

Every principal G-bundle over M^3 is trivial; fix a trivialization $G \to P \to M$ (section $s: M \to P$).

For any connection ω on P, get $CS(\omega) \in \Omega^3(P;\mathbb{R})$.

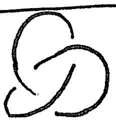
pull back to M along section and integrate:

$$A_{CS}(\omega) = \frac{k}{4\pi}\int_M s^* \operatorname{tr}\left(d\omega\wedge\omega + \tfrac{2}{3}\omega\wedge\omega\wedge\omega\right)$$

gives "action functional" on space of all connections on $P = M\times G$

critical points = flat connections

Not Morse b/c $C^\infty(M,G)$ acts on space of connections preserving A_{CS}

Given a knot K  and a connection ω, can look at monodromy around K.

$$\int e^{iA_{CS}(\omega)} \cdot \operatorname{tr}(\text{monodromy of } \omega \text{ around } K) \quad D\text{connections}$$

Feynman path integral over space of all connections up to gauge equivalence (action of $C^\infty(M,G)$)

turns out to be a polynomial in $q = e^{2\pi i/k+2}$

Theorem (Witten): this is the Jones polynomial of K in M^3.

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dominant terms are critical points of action functional, i.e. flat connections.

up to gauge equivalence, flat connections $\longleftrightarrow$ representations $\pi_1(M^3) \to G$ up to conjugacy

so asymptotics are counting # of representations

can be used to give gauge-theoretic definition of Casson invariant.