Part 4 of Reading Course

Vector fields and derivations

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Notation. If M is a manifold, here is the standard notation:

- $C^\infty(M)$ = the vector space of infinitely-differentiable functions on M
- $\mathfrak{X}(M)$ = the vector space of vector fields on M
- $\Omega^k(M)$ = the vector space of k-covector fields on M (usually called *differential k-forms*)

Note that $\Omega^0(M)$ is the same as $C^\infty(M)$, since a 0-covector field is just a function.

An example of vector fields. Consider $M = \mathbb{R}^2 - \{0\}$. We have coordinates r and θ on M (ignore here that θ is not perfectly defined), which give covector fields dr and $d\theta$. From these we get two vector fields, for which the *standard notation* is $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, as the unique vector fields satisfying

$$dr(\frac{\partial}{\partial r}) = 1 \qquad dr(\frac{\partial}{\partial \theta}) = 0 \qquad\qquad d\theta(\frac{\partial}{\partial r}) = 0 \qquad d\theta(\frac{\partial}{\partial \theta}) = 1$$

(note that 1 here is the constant function 1, and 0 is the constant function 0).

Exercise 1. Sketch the vector fields $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ (like, actually sketch them on paper, with arrows).

Exercise 2. Write $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ in terms of the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ for vector fields.
(i.e. write each as $\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}$ for some functions α and β)

Vector fields as operators. Given a vector field $V \in \mathfrak{X}(M)$, we obtain an operator (i.e. a linear transformation)

$$\partial_V : C^\infty(M) \to C^\infty(M)$$

defined as follows:

Given a function f, we need to define the function $\partial_V f$; its value at a point p is defined as follows. Let V_p be the vector at p determined by the vector field V. Then $(\partial_V f)(p)$ is the directional derivative of f at p in the direction of V_p:

$$(\partial_V f)(p) \coloneqq \lim_{\Delta t \to 0} \frac{f(p + \Delta t \cdot V_p) - f(p)}{\Delta t}$$

Exercise 3. Let $V = \frac{\partial}{\partial r}$ and $W = \frac{\partial}{\partial \theta}$. For each of the following functions f, compute $\partial_V f$ and $\partial_W f$:
(a) $f = e^{x^2+y^2}$; (b) $f = e^x$.

Exercise 4. Check that if $V = \frac{\partial}{\partial r}$ and $W = \frac{\partial}{\partial \theta}$, then $\partial_V(\partial_W f) = \partial_W(\partial_V f)$ for all functions f.
In other words, the operators ∂_V and ∂_W *commute.*

Exercise 5. Exhibit two vector fields V and U on $\mathbb{R}^2$ (or on some subset of $\mathbb{R}^2$) for which ∂_V and ∂_U do *not* commute.

Exercise 6. Very hard, most important, spend most of your time on this: Let $\Phi: C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ be a linear transformation satisfying the "product rule" $\Phi(fg) = \Phi(f)g + f\Phi(g)$ for all functions $f, g \in C^\infty(\mathbb{R}^2)$. Prove that there exists a unique vector field V on $\mathbb{R}^2$ such that $\Phi = \partial_V$.

[Hint: if you like, you can first do this on $\mathbb{R}^1$, where the question becomes: Let $\Phi: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be a linear transformation satisfying the "product rule" $\Phi(fg) = \Phi(f)g + f\Phi(g)$ for all functions $f, g \in C^\infty(\mathbb{R})$. Prove that there exists a unique function u such that $\Phi(f) = u \cdot f'$ for all f.]