Part 3 of Reading Course

Introduction to de Rham cohomology

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In Part 2 we learned about 2-covector fields. In particular, we learned about the operator $d\colon \{\text{1-covector fields}\} \to \{\text{2-covector fields}\}$. In this part we will start with a *different* way of combining 1-covector fields to obtain a 2-covector field.

Recall from [linear algebra class] that if v and w are elements of V^* then $v \wedge w$ is an element of $\bigwedge^2 V^*$; that is, if v and w are 1-covectors, then $v \wedge w$ is a 2-covector (of course, it's possible that $v \wedge w$ might be 0).

If ω and η are 1-covector fields, then applying the above operation at each point will give us a 2-covector field $\omega \wedge \eta$. This obeys the same rules as in [linear algebra class], such as

$$\omega \wedge \omega = 0 \qquad \text{and} \qquad \eta \wedge \omega = -(\omega \wedge \eta).$$

In particular, considering the 1-covector fields dx and dy, we have $dx \wedge dx = 0$ and $dy \wedge dy = 0$ and $dy \wedge dx = -(dx \wedge dy)$.

Say we are on $\mathbb{R}^2$, so that we can write the 1-covector fields ω and η as $\omega = \alpha\, dx + \beta\, dy$ and $\eta = \gamma\, dx + \delta\, dy$. Then we can compute:

$$\begin{aligned}
\omega \wedge \eta &= (\alpha\, dx + \beta\, dy) \wedge (\gamma\, dx + \delta\, dy) \\
&= \alpha\, dx \wedge \gamma\, dx + \alpha\, dx \wedge \delta\, dy + \beta\, dy \wedge \gamma\, dx + \beta\, dy \wedge \delta\, dy \\
&= (\alpha\gamma)\, dx \wedge dx + (\alpha\delta)\, dx \wedge dy + (\beta\gamma)\, dy \wedge dx + (\beta\delta)\, dy \wedge dy \\
&= (\alpha\gamma) \cdot 0 + (\alpha\delta)\, dx \wedge dy - (\beta\gamma)\, dx \wedge dy + (\beta\delta) \cdot 0 \\
&= (\alpha\delta - \beta\gamma)\, dx \wedge dy
\end{aligned}$$

Let $M = \mathbb{R}^2 - \{0\}$ be the plane excluding the origin. Consider the functions r and θ on M defined by $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(\frac{y}{x})$. Conversely, we have $x = r\cos\theta$ and $y = r\sin\theta$.

Exercise 1. Convince yourself that $d\theta$ is really a well-defined 1-covector field on M, even though θ itself is not so well-defined (does it take values in $[0, 2\pi]$? or $[-\pi, \pi]$? or $[0, 2\pi)$? etc).

Exercise 2. Show that at every point of M the covectors given by dr and $d\theta$ at that point give a basis for the covectors there.
Conclude that every covector field on M can be written uniquely as $\zeta dr + \xi d\theta$ for functions ζ and ξ.

Exercise 3. Find a function λ so that $dx \wedge dy = \lambda \cdot (dr \wedge d\theta)$.

Exercise 4. Find a function μ so that $dr \wedge d\theta = \mu \cdot (dx \wedge dy)$.

Exercise 5. Do you recognize the functions λ and μ from anywhere? How are they related to each other?

(The operators discussed below exist on any manifold, but let's continue to assume we're on the plane $\mathbb{R}^2$.)

Recall from Part 2 that we defined a operator $d\colon \{\text{1-covector fields}\} \to \{\text{2-covector fields}\}$ by:

$$d(\alpha\, dx + \beta\, dy) \quad = \quad \Big(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\Big)\, dx \wedge dy \tag{1}$$

And of course in Part 1 we defined the operator $d\colon \{\text{functions}\} \to \{\text{1-covector fields}\}$ by:

$$df \quad = \quad \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$$

Exercise 6. Let f be any function and let ω be any 1-covector field ω. We can multiply these to get a new 1-covector field $f\omega$. Show that we have an equality of 2-covector fields

$$d(f\omega) = df \wedge \omega + f \wedge d\omega. \tag{2}$$

To prove (2), you must have used the definition (1). But actually (2) is the more basic property, because it doesn't depend on a choice of coordinates. And it turns out we could instead have *started* with (2) and deduced (1) from it, as you'll show in the following exercise.

Exercise 7. Assume that $d\colon \{\text{1-covector fields}\} \to \{\text{2-covector fields}\}$ satisfies the properties

$$d(f\omega) = df \wedge \omega + f \wedge d\omega, \qquad d(dx) = 0, \qquad d(dy) = 0.$$

Without using the definition (1), *prove* from these assumptions that $d(\alpha\, dx + \beta\, dy) = \big(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\big)\, dx \wedge dy$.

Exercise 8. Now say we are on $\mathbb{R}^3$, where $df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz$. Note that on $\mathbb{R}^3$, any 1-covector field ω can be written as $d(\alpha\, dx + \beta\, dy + \gamma\, dz)$.

Assume that $d\colon \{\text{1-covector fields}\} \to \{\text{2-covector fields}\}$ satisfies the properties (for any function f and any 1-covector field ω):

$$d(f\omega) = df \wedge \omega + f \wedge d\omega, \qquad d(dx) = 0, \qquad d(dy) = 0, \qquad d(dz) = 0.$$

Use these assumptions to find a formula for $d(\omega)$ (in terms of α, β, γ).

Exercise 9. Finally (still on $\mathbb{R}^3$) we would like to define an operator $d\colon \{\text{2-covector fields}\} \to \{\text{3-covector fields}\}$ satisfying the properties (for any function f and any 2-covector field φ):

$$d(f\varphi) = df \wedge \varphi + f \wedge d\varphi, \qquad d(dx \wedge dy) = 0, \qquad d(dy \wedge dz) = 0, \qquad d(dx \wedge dz) = 0.$$

From these assumptions, find a formula for $d\varphi$ for any 2-covector field φ.

(Hint: first, find a way to write the 2-covector field φ in coordinates, analogous to how we could write a 1-covector field ω as $\omega = \alpha\, dx + \beta\, dy + \gamma\, dz$.)

Exercise 10. You defined an operator $d\colon \{\text{2-covector fields}\} \to \{\text{3-covector fields}\}$ in the previous exercise. (We are still on $\mathbb{R}^3$.) Show that for any 1-covector fields ω and η, this operator satisfies

$$d(\omega \wedge \eta) = d\omega \wedge \eta - \omega \wedge d\eta. \tag{3}$$

(This is just like the equation (2), except that in (3) we have a $-$ instead of a $+$.)

After doing these exercises, read the first 3 pages of Bott–Tu §1.

[Very helpful if you have access to Bott–Tu, but not essential if not.]