# Part 1 of Reading Course

## Introduction to de Rham cohomology

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[If you have Eliashberg's notes from Math 52H/62CM, read §7.1 and §7.4.]

We are interested in understanding the operator $d$, which takes a function $f$ on $\mathbb{R}^n$, and produces a covector field[1] $df$ on $\mathbb{R}^n$. In general, if $f$ is a function on any manifold $M$, then $df$ will be a covector field on $M$; but for now we stick with the case $M = \mathbb{R}^n$.

**Covector fields.** In fact, let us work with $\mathbb{R}^2$, since we can see everything important there. The standard coordinates give us 2 "basic" covector fields on $\mathbb{R}^2$, called $dx$ and $dy$. The first is the covector field which (at each point) sends a vector to its $x$-coordinate, and the second is the covector field which (at each point) sends a vector to its $y$-coordinate.

Every covector field $\omega$ on $\mathbb{R}^2$ can be written uniquely as $\omega = \alpha\, dx + \beta\, dy$ for unique functions $\alpha$ and $\beta$ on $\mathbb{R}^2$. (Think about why this is!)

**The operator $d$.** We can define the operator $d$ very simply: given a function $f$ on $\mathbb{R}^2$, we define

$$df \stackrel{\text{def}}{=} \frac{\partial f}{\partial x}\, dx \quad + \quad \frac{\partial f}{\partial y}\, dy$$

For example, for the function $f(x, y) = xy^2$ on $\mathbb{R}^2$, we have $df = y^2\, dx + 2xy\, dy$. [If you have Eliashberg's notes, compare with §7.1 for a conceptual explanation of $df$ as a linear approximation.]

On $\mathbb{R}^1$ this is even simpler: given a function $f$ on $\mathbb{R}^1$, we have just

$$df \stackrel{\text{def}}{=} \frac{\partial f}{\partial x}\, dx$$

**Exercises.**

1. (Warm-up) For $f(x, y) = e^{x+y^2}$, what is $df$? For $g(x, y) = 2x + 3y$, what is $df$?
2. Let $\omega$ be the covector field $\omega = x\, dx + y\, dy$ on $\mathbb{R}^2$. Find a function $f$ on $\mathbb{R}^2$ such that $df = \omega$.
3. Let $\omega$ be an arbitrary covector field on $\mathbb{R}^1$. Prove that there exists some function $f$ on $\mathbb{R}^1$ such that $df = \omega$. (Hint: write $\omega = \alpha\, dx$ for some function $\alpha$.)
4. In contrast with Exercise 3: give an example of a covector field $\omega = \alpha\, dx + \beta\, dy$ on $\mathbb{R}^2$ for which there **cannot exist any** function $f$ with $df = \omega$.
5. (Challenge) Let $\omega = \alpha\, dx + \beta\, dy$ be a covector field on $\mathbb{R}^2$. Describe precise conditions on $\alpha$ and $\beta$ that guarantee that there *does* exist a function $f$ with $df = \omega$.
   (Challenge) When these conditions are satisfied, how can we actually *find* the function $f$?
6. (Challenge) Let $C$ be the unit circle. In contrast with Exercise 3, describe a covector field $\omega$ on the unit circle $C$ for where there does not exist any function $f$ on $C$ with $df = \omega$.
   [This will require thinking about how you want to describe a covector field on $C$.]

[1] A vector field is a choice, at each point of $\mathbb{R}^n$, of a vector there; similarly a covector field is a choice, at each point of $\mathbb{R}^n$, of a *linear function from vectors there to* $\mathbb{R}$. (Eventually we will use the term "differential 1-form" instead of "covector field".)