Geometric Literacy 2/22/10

Chern-Weil theory

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Examples:

1) Given $\begin{array}{c} \mathbb{R}^n \to E \\ \downarrow \\ M \end{array}$, frame bundle is $\begin{array}{c} GL_n\mathbb{R} \to FE \\ \downarrow \\ M \end{array}$

(over each $p \in M$, set of bases for fiber)

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ acts on frames by $(v_1, v_2) \mapsto (2v_1 + v_2, v_1 + v_2)$

2) Circle bundle with arclength element $d\theta$ on each fiber

S^1 acts by rigidly rotating each fiber

G a Lie group, $\mathfrak{g} = T_eG$ its Lie algebra

$G \circlearrowright G$ on the left, identifies $\mathfrak{g}$ with left-invariant vector fields

$x \in \mathfrak{g} \longleftrightarrow L_x$ vector field

Lie bracket on vector fields gives

$L_{[x,y]} := [L_x, L_y]$

Principal G-bundle is a bundle with all fibers G

is a space $\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ M \end{array}$ with charts $\begin{array}{c} U \times G \\ \downarrow \\ U \end{array}$ so that

$(u, h) \cdot g = (u, hg)$

Proposition: $\begin{array}{c} G \to P \\ \downarrow \\ M \end{array}$ trivial iff it has a section $\begin{array}{c} P \\ \uparrow s \\ M \end{array}$

Proof: $M \times G \to P$ is an isomorphism

$(m, g) \longmapsto s(m) \cdot g$

Maurer-Cartan form (tautological form)

$\omega_0 \in \Omega^1(G;\mathfrak{g})$ $\mathfrak{g}$-valued 1-form on G

given $v \in T_pG$, left-translation to identity gives some $x \in T_eG = \mathfrak{g}$

takes in vectors, gives elts of $\mathfrak{g}$ = $\mathfrak{g}$-valued 1-form

left-invariant; on right transforms as $Ad(g^{-1})\omega_0$

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given $\eta \in \Omega^1(G;\mathfrak{g})$, can differentiate as usual, $d\eta \in \Omega^2(G,\mathfrak{g})$

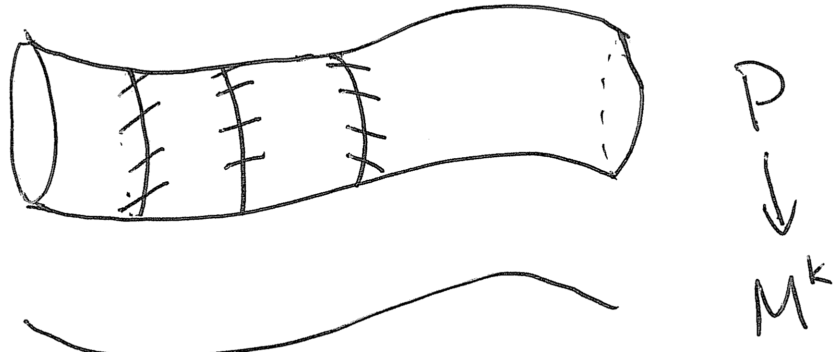
$\omega \wedge \eta \in \Omega^2(G;\mathfrak{g}\wedge\mathfrak{g})$; compose with bracket to get $[\omega,\eta] \in \Omega^2(G;\mathfrak{g})$

Maurer-Cartan equation:

$$d\omega_0 + \tfrac{1}{2}[\omega_0,\omega_0] = 0$$

(revisit + prove next time)

Connections on principal bundles



connection = k-dim. distribution H on P
transverse to fibers
+ invariant under $P \circlearrowleft G$

splitting $T_pP = T_p^{vert} \oplus H$ gives projection $T_pP \to T_p^{vert}$

but fiber is identified with G up to $G \circlearrowleft G$, giving $T_p^{vert} \simeq \mathfrak{g}$

So this projection takes vectors in P, gives element of $\mathfrak{g}$

$\mathfrak{g}$-valued 1-form $\underset{\text{connection}}{\underline{\omega \in \Omega^1(P;\mathfrak{g})}}$, restricts to fibers as ω_0, transforms as $Ad(g^{-1})\omega$

__Ex__: on circle bundle, 1-form restricting to $d\theta$ on each fiber

Curvature

$d\omega + \frac{1}{2}[\omega, \omega] = \Omega$ $\quad \Omega \in \Omega^2(P; \mathfrak{g})$

bundle is flat $\iff \Omega = 0$

(locally trivial even with connection)

Chern-Weil theory

every $\lambda: \mathfrak{g} \to \mathbb{R}$ gives $\underline{\lambda} \in \Omega^1(P; \mathbb{R})$

$\lambda \in \mathfrak{g}^*$ by $\underline{\lambda} = \lambda \circ \omega$

$\lambda \wedge \mu \in \Lambda^2 \mathfrak{g}^* \mapsto \underline{\lambda} \wedge \underline{\mu}$ $\quad \underline{\lambda} \wedge \underline{\mu} = -\underline{\lambda} \wedge \underline{\mu}$

separately, $\lambda \in \mathfrak{g}^*$ gives $\bar{\lambda} \in \Omega^2(P; \mathbb{R})$

by $\bar{\lambda} = \lambda \circ \Omega$ $\quad \bar{\lambda} \wedge \bar{\mu} = \bar{\mu} \wedge \bar{\lambda}$

Choice of connection gives map

$\Lambda^* \mathfrak{g}^* \otimes \mathrm{Sym}^* \mathfrak{g}^* \to \Omega^*(P; \mathbb{R})$

Want to know: when does resulting form on P descend to M?

Answer doesn't depend on connection, only on $\mathfrak{g}$.

Answer: nothing involving $\Lambda^* \mathfrak{g}^*$ factor,

and those elements of $\mathrm{Sym}^* \mathfrak{g}^*$ that are $\mathrm{Ad}(G)$-invariant

Turn out to all be closed as well. Doesn't depend.

Thus each invariant poly $f \in (\mathrm{Sym}^* \mathfrak{g}^*)^{\mathrm{Ad}(G)}$

gives, on any principal bundle $\begin{array}{c} G \to P \\ \downarrow \\ M \end{array}$, a characteristic class $[f] \in H^*(M; \mathbb{R})$

$GL_n\mathbb{R} \leftrightarrow \mathbb{R}[\sigma_1, \ldots, \sigma_n]$

$GL_n\mathbb{C} \leftrightarrow \mathbb{R}[c_1, \ldots, c_n]$

$U(n) \leftrightarrow \mathbb{R}[c_1, \ldots, c_n]$

$O(n) \leftrightarrow \mathbb{R}[p_1, \ldots, p_{\lfloor n/2 \rfloor}]$

$SO(2n) \leftrightarrow \mathbb{R}[p_1, \ldots, p_{n-1}, eu]$

$SO(2n+1) \leftrightarrow \mathbb{R}[p_1, \ldots, p_n]$

for compact G, all characteristic classes arise this way

Answers: why only even dim?