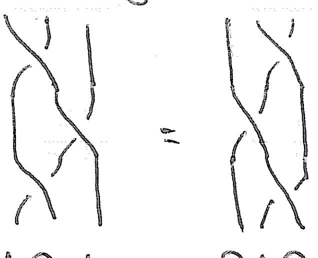
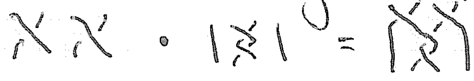
# Definitions, background

Braid group: $B_n :=$ $n$ strands montonically embedded in $\mathbb{C} \times I$
$= \{1, 2, \ldots, n\}$ on top + bottom
modulo time-preserving isotopy



ABA = BAB
w/ $A = \sigma_i$, $B = \sigma_{i+1}$
"braid relation"

Composition: 
Identity: | | | |
Generators: | | X | |  ($\sigma_i$)    Proof: avoid triple points

Theorem (Artin): $B_n = \left\langle \sigma_i \;\middle|\; \begin{array}{l} \sigma_i \text{ and } \sigma_{i+1} \text{ braid} \\ \sigma_i \text{ and } \sigma_j \text{ commute, } |i-j|>1 \end{array} \right\rangle$

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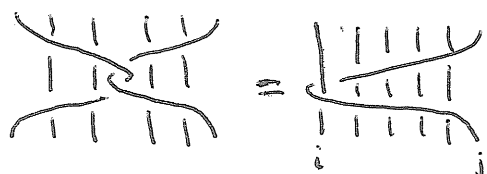
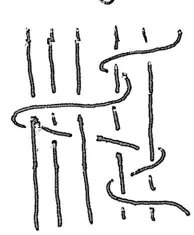
By following strands: $1 \to P_n \to B_n \to S_n \to 1$
$\sigma_i \longmapsto (i \ i{+}1)$

$P_n$ "pure braid group" generated by  $T_{i,j}$

Proof 1: $S_n$ has presentation $\left\langle s_i = (i \ i{+}1) \;\middle|\; \begin{array}{l} s_i^2 = 1 \\ s_i \text{ and } s_{i+1} \text{ braid} \\ s_i \text{ and } s_j \text{ commute} \end{array} \right\rangle$ (Coxeter group)

General lemma:
Given $1 \to K \to G \xrightarrow{\pi} H \to 1$, $g_i$ generators for $G$, and a presentation of $H$ in terms of $\pi(g_i)$
$K$ is normally generated in $G$ by those relations $R_j$ that <u>don't</u> lift to $G$.
Proof: $G/\langle R_j \rangle$ is a presentation for $H$.
So lifting $s_i$ to $\sigma_i$, we find $P_n$ is generated by the $B_n$-conjugates of $\sigma_i^2 = T_{i,i+1}$.
Check that $T_{i,j}$ generate all conjugates. ∎

Proof 2. Forgetting last strand gives

kernel: 

$1 \to F_n \to P_{n+1} \to P_n \to 1$

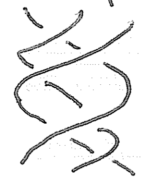
$P_{n+1} = F_n \rtimes P_n$

obvious section $P_n \to P_{n+1}$ (add strand)

By induction, $P_n$ generated by $T_{i,j}$
$F_n$ generated by $T_{i,n+1}$. ∎

$P_{n+1} \cong F_n \rtimes P_n$ yields presentation for $P_n$ inductively.

Configuration spaces:

Consider  as a movie $\underset{t=1}{\overset{t=0}{\downarrow}}$, we see $n$ points moving in the plane.

Each strand traces out $\gamma_i(t)$, w/ $\gamma_i(t) \neq \gamma_j(t)$ and $\gamma_i(0)=\gamma_i(1)=i$ (pure braid).

Key definition: $X_n := \mathbb{C}^n \setminus \bar{\Delta} = \mathbb{C}^n \setminus \{\text{any } z_i = z_j\}$

$\bar{p} = (1,\dots,n)$

a pure braid $\rightsquigarrow$ an $n$-strand movie $\rightsquigarrow$ $n$ loops $\gamma_i$ $\rightsquigarrow$ one loop $\bar{\gamma}$ in $X_n$

$\bar{\gamma}(0)=\bar{\gamma}(1)=\bar{p}$ level-preserving isotopy $\longleftrightarrow$ homotopy of $\bar{\gamma}$

Thus: $P_n = \pi_1(X_n, \bar{p})$

If $Y_n = X_n/S_n$ is the unordered configuration space, $B_n = \pi_1(Y_n, \bar{p})$

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Cohomology groups of $P_n$

Natural projection $X_{n+1} \to X_n$, fiber over $\bar{p}=(1,\dots,n)$ is $\mathbb{C}\setminus\{1,\dots,n\}$, is a fiber bundle.

Corollary: $X_n$ is a $K(P_n,1)$. Proof: LES in homotopy shows $F \to E \to B$ if $F$ and $B$ are aspherical, so is $E$. Induction on $n$.

Claim: $H^*(P_n) \approx H^*(\mathbb{Z}) \otimes H^*(F_2) \otimes H^*(F_3) \otimes \cdots \otimes H^*(F_{n-1})$ <u>as abelian groups.</u>

$= H^*(\mathbb{Z} \times F_2 \times F_3 \times \cdots \times F_{n-1})$

Proof: Consider Leray-Serre spectral sequence for $\mathbb{C}\setminus\{1,\dots,n\} \to X_{n+1} \to X_n$.

Note we have a section (explicitly $(z_i) \longmapsto (z_i, \frac{z_1+\cdots+z_n}{n} + 2\cdot\max|z_i - z_j| + 1)$) corresponding to splitting $P_{n+1} = F_n \rtimes P_n$.

Since $H^*(\mathbb{C}\setminus\{1,\dots,n\})$ is only $H^0=\mathbb{Z}$, $H^1=\mathbb{Z}^n$, and action of $P_n$ is trivial (pure braid group),

$E_2$ page looks like

$(E_2^{p,q} = H^p(B;H^q(F))$

$$\begin{matrix} H^0(X_n)\otimes\mathbb{Z}^n & H^1(X_n)\otimes\mathbb{Z}^n & H^2(X_n)\otimes\mathbb{Z}^n & \cdots & H^n(X_n)\otimes\mathbb{Z}^n \\ H^0(X_n) & H^1(X_n) & H^2(X_n) & \cdots & H^n(X_n) \end{matrix}$$

(same as for direct product $F_n \times P_n$)

Existence of section $X_n \to X_{n+1}$ implies $d_2$ is zero (black box), and all later differentials are 0, so $E_2 = E_3 = \cdots = E_\infty$. By induction, everything is free abelian, so no extension problems. ■

Corollary: $H^*(P_n)$ is torsion-free.

# Cup product in $H^*(P_n)$

Generators for $H^1(P_n) \cong \mathrm{Hom}(P_n, \mathbb{Z})$ given by

$W_{ij} : P_n \to \mathbb{Z}$ measuring winding of $i^{th}$ strand around $j^{th}$

$W_{ij}(T_{ij}) = 1$ , $W_{ij}(T_{\cdot\cdot}) = 0$

Fundamental relation:

$$W_{ij} \wedge W_{jk} + W_{jk} \wedge W_{ki} + W_{ki} \wedge W_{ij} = 0 \quad \text{in } H^2(P_n) \qquad (\text{call this } R_{ijk} = 0)$$

Corollary: $H^*(P_n) = \Lambda^*[W_{ij}] / (R_{ijk})$.

Proof: spectral sequence implies $H^*(P_n)$ generated by $W_{ij}$, so the natural map $\Lambda^*[W_{ij}]/(R_{ijk}) \to H^*(P_n)$ is onto. Calculation shows ranks are the same, thus isomorphism.

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# 3 proofs of fundamental relation (de Rham, bar cochain, Sullivan theory)

## Proof 1:

Define $\omega_{ij} = \dfrac{dz_i - dz_j}{z_i - z_j} \in \Omega^1(X_n; \mathbb{C})$.

Note $\omega_{ij} = d\big(\log(z_i - z_j)\big)$ so $\omega_{ij}$ is closed;

Cauchy integral formula $\Rightarrow \oint_\gamma \omega_{ij}$ = winding number of $\gamma_i(t)$ around $\gamma_j(t)$

so $\omega_{ij}$ represents $W_{ij} \in H^1(X_n; \mathbb{C})$.

Claim: the 2-form below is identically 0 at every point:

$$P_{ijk} = \left(\frac{dz_i - dz_j}{z_i - z_j}\right) \wedge \left(\frac{dz_j - dz_k}{z_j - z_k}\right) + \left(\frac{dz_j - dz_k}{z_j - z_k}\right) \wedge \left(\frac{dz_k - dz_i}{z_k - z_i}\right) + \left(\frac{dz_k - dz_i}{z_k - z_i}\right) \wedge \left(\frac{dz_i - dz_j}{z_i - z_j}\right)$$

Proof of claim: would suffice to expand and collect terms.

We can set $dz_i - dz_j = A$, $z_i - z_j = a$, etc, so this is $\dfrac{A \wedge B}{ab} + \dfrac{B \wedge C}{bc} + \dfrac{C \wedge A}{ca}$.

Clear denominators and note $A + B + C = 0$, $a + b + c = 0$.

$$\begin{aligned} abc\,P_{ijk} &= cA \wedge B + aB \wedge C + bC \wedge A \\ &= (-a-b)A \wedge B + aB \wedge (-A-B) + b(-A-B) \wedge A \\ &= (-a-b+a+b)A \wedge B \\ &= 0. \; \blacksquare \end{aligned}$$

Thus $R_{ijk} = 0$ in $H^2(P_n; \mathbb{C})$. Since $H^*(P_n)$ is torsion-free, $R_{ijk} = 0$ in $H^2(P_n)$. $\blacksquare$

Corollary to claim: $W_{ij} \mapsto \omega_{ij}$ defines an injection

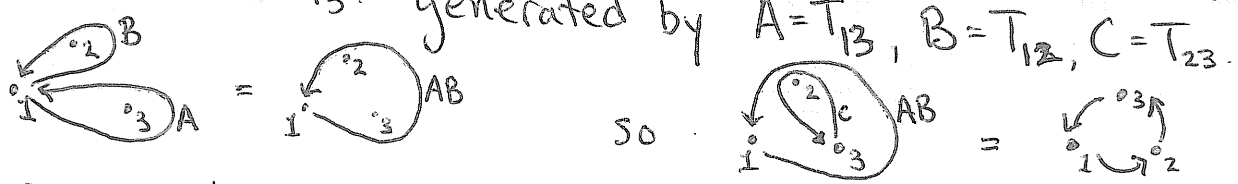
$$H^*(P_n) \hookrightarrow \Omega^*(X_n; \mathbb{C}) \text{ into the algebra of closed differential forms.}$$

Corollary: $X_n$ is formal (in the sense of Sullivan theory).

Proof 2: our relation only involves 3 strands; should be "supported on smaller subgroup".

Forgetting all but strands $i,j,k$ gives map $P_n \to P_3$;
since $W_{ij}$ pulls back to $W_{ij}$, suffices to show $R_{123} = 0$ in $H^2(P_3)$.

Let's understand $P_3$: generated by $A = T_{13}$, $B = T_{12}$, $C = T_{23}$.




rotation through $2\pi$

turn your head by $\frac{2\pi}{3}$ and then by $\frac{4\pi}{3}$, and same argument shows
$ABC = BCA = CAB$ = rotation through $2\pi$ (also notice that $AB$ and $C$ commute).

Lemma: $P_3 = \langle A,B,C \mid ABC = BCA = CAB \rangle$.

Proof: Can derive from $P_3 = F_2 \rtimes \mathbb{Z} = \langle A, C\rangle \rtimes \langle B \rangle$.

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Idea of proof: pretend $P_3 = F_3$, then show dreams come true.

Let $\alpha, \beta, \gamma : P_3 \to \mathbb{Z}$ be dual to $A,B,C$. Pull back to $F_3 = \langle A,B,C\rangle \to P_3$.

Certainly $\alpha\wedge\beta + \beta\wedge\gamma + \gamma\wedge\alpha = 0$ in $H^2(F_3)$ b/c $H^2(F_3) = 0$; so there is some cochain
$\varphi \in C^1(F_3;\mathbb{Z})$ = Set-maps $F_3 \to \mathbb{Z}$ s.t. $\delta\varphi = \alpha\wedge\beta + \beta\wedge\gamma + \gamma\wedge\alpha$.

We will show that $\varphi$ descends to $C^1(P_3;\mathbb{Z})$ so that $R_{123} = \delta\varphi = 0$ in $H^2(P_3)$.

<u>Simpler example</u>: $\alpha\wedge\beta \in C^2(F_3;\mathbb{Z})$ maps $(g_1,g_2) \longmapsto \alpha(g_1)\cdot\beta(g_2)$.

We want $f : F_3 \to \mathbb{Z}$ so that $f(g_1 g_2) = f(g_1) + f(g_2) + \alpha(g_1)\beta(g_2)$ (✱) $[\delta f = \alpha\wedge\beta]$.

Can normalize so that $f(A) = f(B) = f(C) = 0$ by adding elements of $Z^1(F_3;\mathbb{Z}) = \mathrm{Hom}(F_3;\mathbb{Z})$

Then (✱) implies $f(BA) = 0$, $f(AB) = 1$, $f(BAB) = 1$, $f(ABAB) = 3$; we see that $f$ counts how many $A$s are to the left of how many $B$s.

For any $g = g_1 \cdots g_k$, $f(g) = \sum_{i<j} \alpha(g_i)\beta(g_j)$; if well-defined, this determines $f$ uniquely.

This formula does not give a well-defined map in $C^1(P_3;\mathbb{Z})$,
because we would have $f(ABC) = 1 \neq 0 = f(BCA)$. (which is good, because $\alpha\wedge\beta \neq 0$ in $H^2(P_3)$).

But $\varphi$, which counts $A$s to the left of $B$s, plus $B$s to the left of $C$s, plus $C$s to the left of $A$s, is well-defined,
because replacing $ABC$ with $BCA$ or $CAB$ leaves $\varphi$ constant. (Exercise.)

Key observation: $\delta\varphi = \alpha\wedge\beta + \beta\wedge\gamma + \gamma\wedge\alpha \in C^2(P_3;\mathbb{Z})$. (Easier to see $\delta f = \alpha\wedge\beta$.)

Thus $R_{123} = \delta\varphi$ and thus is $0$ in $H^2(P_3)$.

Proof 3:
# Commutators and cup product

$[\ ,\ ]: G\times G \to [G,G]$ set map $[x,yz]=[x,y]\,y[x,z]y^{-1}$

$(x,y)\longmapsto [x,y]$

since $[G,G]\times G$, $G\times[G,G]$ map to $[[G,G],G]$, descends to bilinear map

$[\ ,\ ]: H_1(G)\wedge H_1(G) \to [G,G]/[[G,G],G]$ always onto

Theorem (Sullivan, Chen)

$$0\to\left([G,G]/[[G,G],G]\right)^* \xrightarrow{[\ ,\ ]^*} H^1(G)\wedge H^1(G) \xrightarrow{\wedge} H^2(G) \quad \text{is exact.}$$

Examples:

- $G=\mathbb{Z}^2$

$$0\to 0\to \Lambda^2\mathbb{Z}^2 \xrightarrow{\approx} \mathbb{Z}$$

- $G=\pi_1(\Sigma_g)=\langle a_i,b_i \mid [a_1,b_1]\cdots[a_g,b_g]=1\rangle$ $\quad H_1(G)=\langle A_i,B_i\rangle$ $\quad H^1(G)=\langle \alpha_i,\beta_i\rangle$

$$0\to\left(\Lambda^2H_1(G)/A_1\wedge B_1+\cdots+A_g\wedge B_g\right) \xrightarrow{[\ ,\ ]^*} \Lambda^2H^1(G) \xrightarrow{\omega} \mathbb{Z}$$

$\omega$ symplectic form

$\alpha_i\wedge\beta_i \longmapsto 1$

$\left.\begin{matrix}\alpha_i\wedge\alpha_j\\ \beta_i\wedge\beta_j\\ \alpha_i\wedge\beta_j\end{matrix}\right]\longmapsto 0$

$\operatorname{im}[\ ,\ ]^*=\{\lambda\in\Lambda^2H^1(G)\mid \lambda(A_1\wedge B_1+\cdots+A_g\wedge B_g)=0\}=\ker\omega$

- Heisenberg group $N=\langle x,y,z \mid [x,y]=z,\ [x,z]=[y,z]=1\rangle$

$$0\to\mathbb{Z}\xrightarrow{\approx}\Lambda^2\mathbb{Z}^2\to\mathbb{Z}^2$$

thus the cup product $H^1(N)\wedge H^1(N)\to H^2(N)$ is trivial

pure braid group $P_3=\langle a,b,c \mid abc=bca=cab\rangle$ $\quad H_1(P_3)=\langle A,B,C\rangle$ $\quad$ Note $[ab,c]=1$, $[bc,a]=1$, $[ca,b]=1$,

and thus $[P_3,P_3]/[[P_3,P_3],P_3] = \Lambda^2H_1(P_3)/\langle (A+B)\wedge C,\ (B+C)\wedge A,\ (C+A)\wedge B\rangle$

It follows that $\operatorname{im}[\ ,\ ]^*=\{\lambda\in\Lambda^2H^1(P_3)\mid \lambda((A+B)\wedge C)=0,\ \lambda((B+C)\wedge A)=0,\ \lambda((C+A)\wedge B)=0\}$

$=\{\lambda\in\Lambda^2H^1(P_3)\mid \lambda(A\wedge B)=\lambda(B\wedge C)=\lambda(C\wedge A)\}$

$=\langle \alpha\wedge\beta+\beta\wedge\gamma+\gamma\wedge\alpha\rangle$,

and the kernel of $H^1(P_3)\wedge H^1(P_3)\to H^2(P_3)$ is spanned by $\alpha\wedge\beta+\beta\wedge\gamma+\gamma\wedge\alpha=R_{123}$. ∎

# Stein spaces

We calculated $H^{\bullet}(P_n)$ and found that $H^i(P_n)=0$, $i \geq n$.
We could have done almost as well without calculation:

A Stein space is a complex manifold satisfying certain vanishing theorems
a good heuristic is that an open domain $\Omega \subset \mathbb{C}^n$ is Stein if for every point $p \in \partial\Omega$, there is a function holomorphic on $\Omega$ and blowing up at $p$.

Examples:

Any domain in $\mathbb{C}$ is Stein.
The ball $B_r = \{(z_i) \in \mathbb{C}^n \mid \Sigma \|z_i\|^2 < 1\}$ is Stein.
The configuration space $X_n$ (indeed any hypersurface complement) is Stein.
Pf: Use the functions $\frac{1}{z_i - z_j}$, holomorphic on $X_n$.

Non-examples:

The annulus $B_2 \setminus \overline{B}_1$ is not Stein.
The punctured space $\mathbb{C}^n \setminus \{0\}$ or $B_r \setminus \{0\}$ is not Stein.

Theorem: A Stein space $X$ of complex dimension $n$ is homotopy equivalent to a CW complex of dimension $n$.

Corollary: $P_n$ has a $K(P_n,1)$ CW complex of dimension $n$.
(Explicit complexes known, e.g. Salvetti complex associated to any hyperplane arrangement)

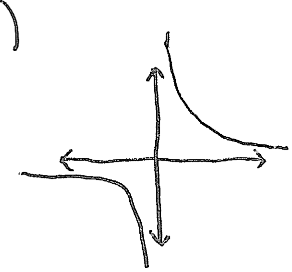
Proof: relies on fact that $X$ is biholomorphic to a complex submanifold closed in $\mathbb{C}^N$.

For example, $X_n \hookrightarrow \mathbb{C}^N$, $N = n + \binom{n}{2}$

$$(z_i) \longmapsto (\ldots z_i \ldots, \ldots \tfrac{1}{z_i - z_j} \ldots)$$

Corresponds to $\mathbb{R} \setminus \{0\} \hookrightarrow \mathbb{R}^2$

$$x \longmapsto (x, \tfrac{1}{x})$$



# Morse theory and focal points

Let $M^k \subset \mathbb{R}^N$ be a real submanifold closed in $\mathbb{R}^N$.

Natural map from normal bundle $NM^k \to \mathbb{R}^N$.

A <u>focal point</u> is a critical value of this map (where normals focus). Rank of a focal point is nullity of Jacobian there.

By Sard's theorem, focal points have measure 0.

**Lemma:** For $q \in M^k$ and $\vec{v}$ normal to $M^k$ at $q$, let $K_1, \dots, K_n$ be the principal ~~radii of~~ curvatures of $M$ at $q$ in the direction of $\vec{v}$.
Then the focal points on the line $q + t\vec{v}$ occur at $q + K_i^{-1}\vec{v}$ (with multiplicity).

**Corollary:** there are at most $n$ focal points (with multiplicity) on such a line.

**Theorem:** For $p \in \mathbb{R}^N$, define $L_p : \mathbb{R}^N \to \mathbb{R}$ by $L_p(q) = \|p - q\|^2$. Restrict to $M^k$.
Then
1) $\vec{q}$ is a critical point of $L_p \Leftrightarrow$ the line from $q$ to $p$ is normal to $M$.
2) $q$ is a degenerate critical point $\Leftrightarrow$ $p$ is a focal point (Corollary: $L_p$ is Morse for almost all $p$)
3) the index of $q$ is the number of focal points (with multiplicity) on the line from $q$ to $p$.

**Lemma:** Let $M^{2k} \subset \mathbb{R}^{2N}$ as above be a <u>complex</u> submanifold.
Then the focal points on line $q + t\vec{v}$ are symmetrically distributed; if $q + t\vec{v}$ has multiplicity $m$, $q - t\vec{v}$ is a focal point with same multiplicity.

**Corollary:** in this case, all critical points have index $\leq k$.

Proof: on each line $q + t\vec{v}$ at most $2k$ focal points occur; by symmetry at most $k$ are on one side of $M$, and thus at most $k$ occur on the interval from $q$ to $p$.

**Corollary:** A complex submanifold $M^{2k} \subset \mathbb{R}^{2N}$ closed in $\mathbb{R}^{2N}$ has the homotopy type of a CW complex of dimension $\leq k$.

Cohomology of $B_n$

Recall $Y_n = (\mathbb{C}^n \setminus \bar{\Delta})/S_n$.

Viete map: think of $\mathbb{C}^n = (\lambda_1, \dots, \lambda_n)$, send $(\lambda_1, \dots, \lambda_n)$ to $t^n + z_1 t^{n-1} + \dots + z_n$, the monic polynomial with roots $\lambda_1, \dots, \lambda_n$

Since this doesn't depend on the order of the $\lambda_i$, descends to a map

$$\mathbb{C}^n/S_n \xrightarrow{\approx} \mathbb{C}^n$$
$$(\lambda_1, \dots, \lambda_n) \longmapsto (z_1, \dots, z_n)$$

explictly, $z_1 = \pm(\lambda_1 + \dots + \lambda_n)$
$z_2 = \pm(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \dots)$
$\vdots$
$z_n = \pm \lambda_1\lambda_2 \cdots \lambda_n$

These are the elementary symmetric polynomials;

Fact: any symmetric polynomial in the $\lambda_i$ is a polynomial in the $z_i$.

Algebro-geometrically:

ring of functions on $\mathbb{C}^n$ is $\mathbb{C}[\lambda_1, \dots, \lambda_n]$, and $S_n$ acts on $\mathbb{C}[\lambda_1, \dots, \lambda_n]$

naive definition of $\mathbb{C}^n/S_n$ is as the variety whose ring of functions is the $S_n$-invariant functions $\mathbb{C}[\lambda_1, \dots, \lambda_n]^{S_n}$

Fact, restated: $\mathbb{C}[\lambda_1, \dots, \lambda_n]^{S_n} = \mathbb{C}[z_1, \dots, z_n]$. (this time, naive idea works)

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$\bar{\Delta} \subset \mathbb{C}^n/S_n$ is sent to $D_0 \subset \mathbb{C}^n$, the space of polynomials with repeated roots

$D_0$ is cut out by the polynomial $\prod_{i \neq j} (\lambda_i - \lambda_j)$;

since this is symmetric in the $\lambda_i$, it can be expressed as a polynomial in the $z_i$, called the discriminant

Examples: $n=2$: $(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_1) = -\lambda_1^2 + 2\lambda_1\lambda_2 - \lambda_2^2 = -(\lambda_1+\lambda_2)^2 + 4\lambda_1\lambda_2 = -z_1^2 + 4z_2$ $(b^2 - 4ac)$

$n=3$: $z_1^2 z_2^2 - 4z_2^3 - 4z_1^3 z_3 - 27z_3^2 - 18z_1z_2z_3$

So we need to understand the single hypersurface $D_0 \subset \mathbb{C}^n$.

Compactify to get $S^{2n} = \mathbb{C}^n \cup \{\infty\}$ and $D = D_0 \cup \{\infty\}$

Now Alexander duality gives:

$$H^*(B_n) = H^*(Y_n) = H^*(\mathbb{C}^n \setminus D_0) = H_{2n-*}(S^{2n}, D) = \tilde{H}_{2n-*+1}(D)$$

How do we analyze $D$ and $\tilde{H}_*(D)$?

1) Set $D = D_n$ and consider all $n$ together.
2) Filter $D_n$ by $D_n = D_n^1 \supset D_n^2 \supset \dots$
   where $D_n^k$ is the space of polynomials with at least $k$ doubleroots.
   $[p(t) \in D_n^k \iff \exists\, q(t)$ w/ degree $k$ s.t. $p(t) = (q(t))^2 r(t)]$
3) We have a natural multiplication map $D_n \times D_m \to D_{n+m}$
   $(p_1(t), p_2(t)) \mapsto p_1(t)p_2(t)$

   Since $D_n \times \{\infty\}$ and $\{\infty\} \times D_m$ are mapped to $\infty \in D_{n+m}$,
   this descends to a map $D_n \wedge D_m \to D_{n+m}$. (Recall $A \wedge B = A \times B / {A \times \{*\} \cup \{*\} \times B}$)
   Respects filtration: $D_n^k \wedge D_m^l \to D_{n+m}^{k+l}$.
4) Since each $D_n^k$ is a manifold $\cup \{\infty\}$, can talk about degree of $D_n^k \wedge D_m^l \to D_{n+m}^{k+l}$.
5) "Filtrations" by $n$ and $k$ fit together to give something like a spectral sequence.

Theorems (Arnold):

1) $H^k(B_n) = 0$, $k \geq n$.
2) $H^k(B_n)$ is finite except for $H^0(B_n) = H^1(B_n) = \mathbb{Z}$
3) "stuttering": $H^*(B_{2n+1}) = H^*(B_{2n})$
4) "homological stability": $H^k(B_n) = H^k(B_{2k-2})$, $n \geq 2k-2$.

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$H^*(B_\infty; \mathbb{Z}/p\mathbb{Z})$ known for all $p$ (even as a Hopf algebra: put $B_3$ next to $B_5$, get $B_8$)

$H_*(B_\infty) \simeq H_*(\Omega^2_0 S^2)$: recall $\Omega^2 S^2 = \{\text{maps } S^2 \to S^2, * \to *\}$
components indexed by $\pi_0 \Omega^2 S^2 = \pi_2 S^2 = \mathbb{Z}$ (degree)

instead of $X_n$ = space of $n$ points in the plane, take $\mathcal{D}_n$ = space of disks in the plane

$X_n \simeq \mathcal{D}_n$ and we have a natural map $\mathcal{D}_n \to \Omega^2_n S^2$ by sending everything outside the disks to the basepoint and wrapping each disk around $S^2$ in a standard way.

Fixing an identification $\Omega^2_n S^2 \simeq \Omega^2_0 S^2$, the hard part is to show we get an isomorphism $H_k(\mathcal{D}_n) \simeq H_k(\Omega^2_0 S^2)$ as $n \to \infty$.

The spaces $\mathcal{D}_n$ fit together to give the little 2-disks operad, used by May and Boardman-Vogt in:

Recognition principle for double loop spaces: $X \simeq \Omega^2 Y \iff X$ is an algebra over the little 2-disks operad $\mathcal{D}$.

# Representations of braid groups from cohomology

$B_n$ = homeomorphisms of a disk fixing $\{n \text{ points}\}$ / isotopy fixing those points

So $B_n$ "acts" on $D \setminus \{1, \ldots, n\}$ and acts on $\pi_1(D \setminus \{1, \ldots, n\}) = F_n$
we could look at action on $H^1(D \setminus \{1, \ldots, n\}) = H^1(F_n)$, but $H^1(F_n) = \mathbb{Z}^n$ and this is just the permutation representation $B_n \to S_n$

How about action on $H^1(F_n; V)$ for some nontrivial $F_n$-module $V$?

Not so fast: say $\rho: F_n \to GL(V)$ makes $V$ an $F_n$-module;
then $H^1(F_n; V)$ is represented by crossed homomorphisms
$f: F_n \to V$ satisfying $f(xy) = f(x) + x \cdot f(y)$, where
$x$ acts on $f(y) \in V$ by $\rho(x)$.

Now if $b: F_n \to F_n$, we want to pull back: $b^* f(x) = f(x^b)$

<u>BUT</u> this is no longer a crossed homomorphism for $\rho$; we have $b^* f \in H^1(F_n; V_{b^*\rho})$.

How can we get an action on $H^1(F_n; V)$? What we need is for $b$ to act on $V$ so that the pullback and the action on $V$ cancel each other out.

This is exactly an action of $F_n \rtimes B_n = \langle F, B \mid b^{-1} x b^{*} = x^b \rangle$ on $V$.

Given such an action = a representation $F_n \rtimes B_n \to GL(V)$ and $f \in H^1(F_n; V)$, define ${}^b f \in H^1(F_n; V)$ by ${}^b f(x) = b \cdot f(x^b)$;

note that ${}^b f(xy) = b \cdot f(x^b y^b) = b \cdot f(x^b) + b \cdot x^b \cdot f(y^b) = b \cdot f(x^b) + x \cdot b f(y^b) = {}^b f(x) + x \cdot {}^b f(y)$

So ${}^b f$ really is a crossed homomorphism for the <u>same</u> action on $V$.

But where are we going to get representations of $F_n \rtimes B_n$?

$F_n \rtimes B_n < B_{n+1}$ — subgroup taking last point to itself

One last trick: "coloring" a representation by $t$. Let $t \in \mathbb{C}$ be fixed.
Given $\rho : F_n \rtimes B_n \to GL(V)$, modify to get $\rho_t : F_n \rtimes B_n \to GL(V)$
defined by $\rho_t(x_i) = t \cdot \rho(x_i)$, $\rho_t(b) = \rho(b)$.

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Given $\rho : B_{n+1} \to GL(V)$, restrict to $F_n \rtimes B_n$ and color by $t$;
now we have an action of $B_n$ on $H^1(F_n; V_{\rho_t})$.
This is a new representation $\rho^+ : B_n \to GL(H^1(F_n; V_{\rho_t}))$ — and we can repeat!
Start with $\rho : B_{n+1} \to \mathbb{C}^\times$ the trivial representation,
$\rho^+ : B_n \to GL_n \mathbb{C}$ is the Burau representation (see below).
Start with $\rho : B_n \to GL_n \mathbb{C}$ the Burau representation,
$\rho^+ : B_{n-1} \to GL_{n^2} \mathbb{C}$ is the Lawrence-Krammer representation $\rho_{LK}$.
Theorem (Bigelow, Krammer): $\rho_{LK} : B_{n-1} \to GL_{n^2} \mathbb{C}$ is injective.

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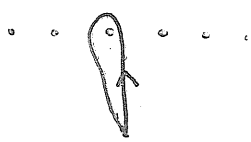
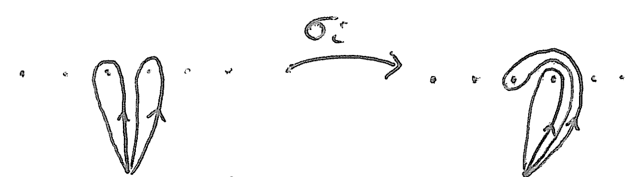
Constructing the Burau representation

$F_n$: 

Action of $B_n$: 

$x_i \longmapsto x_{i+1}$
$x_{i+1} \longmapsto x_{i+1} x_i x_{i+1}^{-1}$

$\rho_t : F_n \rtimes B_n \to \mathbb{C}^\times$
$x_i \longmapsto t$
$b \longmapsto 1$

$Z^1(F_n; \mathbb{C}_t) = \mathbb{C}^n$, with coordinates given by $f(x_i)$
$f \in Z^1(F_n; \mathbb{C}_t) \Rightarrow f(xy) = f(x) + t f(y)$; note $0 = f(x_i x_i^{-1}) = f(x_i) + t f(x_i^{-1})$
$\Rightarrow f(x_i^{-1}) = -t^{-1} f(x_i)$

Let's compute action of $\sigma_i$ on $Z^1(F_n; \mathbb{C}_t)$.

$\sigma_i f(x_i) = f(x_{i+1})$ $\quad$ $\sigma_i f(x_{i+1}) = f(x_{i+1} x_i x_{i+1}^{-1})$ $\quad$ $\sigma_i f(x_j) = f(x_j)$
$= f(x_{i+1}) + t f(x_i) + t^2 f(x_{i+1}^{-1})$
$= f(x_{i+1}) + t f(x_i) - t f(x_{i+1})$

In matrix form:

$$\sigma_i \longmapsto \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1-t & 1 & \\ & & t & 0 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

and this is the standard definition of the Burau representation.