Cohomology of $B_n$

Recall $Y_n = (C^n \setminus \Delta)/S_n$.

Viete map: think of $C^n = (\lambda_1, \ldots, \lambda_n)$, send $(\lambda_1, \ldots, \lambda_n)$ to $z_1^{\lambda_1} + z_2^{\lambda_2} + \cdots + z_n^{\lambda_n}$, the monic polynomial with roots $\lambda_1, \ldots, \lambda_n$.

Since this doesn't depend on the order of the $\lambda_i$, descends to a map

$$C^n/S_n \approx C^n$$

$$\begin{align*}
\lambda_1, \ldots, \lambda_n &\mapsto (z_1, \ldots, z_n) \\
\text{explicitly, } z_1 &= \pm (\lambda_1 + \cdots + \lambda_n) \\
\text{ } z_2 &= \pm (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 + \cdots) \\
\text{ } z_n &= \pm \lambda_1 \lambda_2 \cdots \lambda_n
\end{align*}$$

These are the elementary symmetric polynomials.

Fact: any symmetric polynomial in the $\lambda_i$ is a polynomial in the $z_i$.

Algebro-geometrically:

- Ring of functions on $C^n$ is $C[\lambda_1, \ldots, \lambda_n]$, and $S_n$ acts on $C[\lambda_1, \ldots, \lambda_n]$.
- Naive definition of $C^n/S_n$ is as the variety whose ring of functions is the $S_n$-invariant functions $C[\lambda_1, \ldots, \lambda_n]^{S_n}$.

Fact, restated: $C[\lambda_1, \ldots, \lambda_n]^{S_n} = C[z_1, \ldots, z_n]$. (This time, naive idea works)

$\Delta \subset C^n$ is sent to $D_0 \subset C^n$, the space of polynomials with repeated roots.

$D_0$ is cut out by the polynomial $\prod_{i<j} (\lambda_i - \lambda_j)$.

Since this is symmetric in the $\lambda_i$, it can be expressed as a polynomial in the $z_i$, called the discriminant.

Examples:

- $n = 2$: $(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_1) = -\lambda_1^2 + 2\lambda_1 \lambda_2 - \lambda_2^2 = -(\lambda_1 + \lambda_2)^2 + 4\lambda_1 \lambda_2 = -4z_1^2 + 4z_2 \quad (b^2 - 4ac)$
- $n = 3$: $z_1 z_2^2 - 4z_1 z_3 - 4z_2 z_3 - 27z_3^2 - 18z_1 z_2 z_3$

So we need to understand the single hypersurface $D_0 \subset C^n$. 
Compactify to get \( S^{2n} = C^n \cup \{00\} \) and \( D = D_0 \cup \{00\} \). Now Alexander duality gives:

\[
H^*(B_n) = H^*(Y_n) = H^*(C^n \setminus D_0) = H_{2n-*}(S^{2n},D) = \hat{H}_{2n-*+1}(D)
\]

How do we analyze \( D \) and \( \hat{H}_{*}(D) \)?

1. Set \( D = D_n \) and consider all \( n \) together.
2. Filter \( D_n \) by \( D_n = D_n' \supset D_n^2 \cdots \) where \( D_n^k \) is the space of polynomials with at least \( k \) double roots.
   \[
   \left\{ p(t) \in D_n^k \iff \exists q(t) \text{ w/ degree } k \text{ s.t. } p(t) = (q(t))^{2^k}q'(t) \right\}
   \]
3. We have a natural multiplication map \( D_n \times D_m \to D_{n+m} \) \( (p(t), p_2(t)) \mapsto p(t)p_2(t) \) since \( D_n \times \{00\} \) and \( \{00\} \times D_m \) are mapped to \( \{00\} \in D_{n+m} \) and this descends to a map \( D_n \wedge D_m \to D_{n+m} \). (Recall \( \wedge A \wedge B = A \times B / A^{(k)} \wedge B^{(k)} \)).

4. Respects filtration: \( D_n^k \wedge D_m^k \to D_{n+m}^k \).
5. "Filtrations" by \( n \) and \( k \) fit together to give something like a spectral sequence.

**Theorems (Arnold):**

1. \( H^k(B_n) = 0 \), \( k \neq n \).
2. \( H^*(B_n) \) is finite except for \( H^0(B_n) = H^*(B_n) = \mathbb{Z} \)
3. "stuttering": \( H^*(B_{2n+1}) = H^*(B_{2n}) \)
4. "homological stability": \( H^k(B_n) = H^k(B_{2k-2}) \), \( n > k-2 \).

**H^*(B_{2p}; \mathbb{Z}/p\mathbb{Z})** known for all \( p \) (even as a Hopf algebra: put \( B_3 \) next to \( B_3 \), get \( B_8 \))

\[
H^*_p(B_{2p}) \cong H^*_p(L_0^2 S^2) \quad \text{Recall } L^2 S^2 = \{ \text{maps } S^2 \to S^2, * \to * \} \text{ components indexed by } \pi_0 L_0^2 S^2 = \pi_2 S^2 = \mathbb{Z} \quad \text{(degree)}
\]

Instead of \( X_n = \) space of \( n \) points in the plane, take \( D_n = \) space of disks in the plane \( X_n \cong D_n \) and we have a natural map \( D_n \to L_0^2 S^2 \) by sending everything outside the disks to the basepoint and wrapping each disk around \( S^2 \) in a standard way.

Fixing an identification \( L_0^2 S^2 \cong L_0^2 S^2 \), the hard part is to show we get an isomorphism \( H_k(B_n) \cong H_k(L_0^2 S^2) \) as \( n \to \infty \).

The spaces \( D_n \) fit together to give the little 2-disks operad, used by May and Boardman-Vogt in: Recognition principle: \( X \simeq L^2 Y \iff X \) is an algebra over the little 2-disks operad \( \mathcal{D} \).
Representations of braid groups from cohomology

$B_n = \text{homeomorphisms of a disk fixing } n \text{ points}/\text{isotopy fixing those points}$

so $B_n$ "acts" on $D \setminus \{1, \ldots, n\}$ and acts on $\pi_1(D \setminus \{1, \ldots, n\}) = F_n$

we could look at action on $H'(D \setminus \{1, \ldots, n\}) = H'(F_n)$, but $H'(F_n) = \mathbb{Z}^n$ and this is just

the permutation representation $B_n \to S_n$

How about action on $H'(F_n; V)$ for some nontrivial $F_n$-module $V$?

Not so fast: say $\rho : F_n \to \text{GL}(V)$ makes $V$ an $F_n$-module;

then $H'(F_n; V)$ is represented by crossed homomorphisms

$f : F_n \to V$ satisfying $f(xy) = f(x) + x \cdot f(y)$, where

$x$ acts on $f(y) \in V$ by $\rho(x)$.

Now if $b : F_n \to F_n$, we want to pull back: $b^*f(x) = f(x^b)$

But this is no longer a crossed homomorphism for $\rho$; we have $b^* \in H'(F_n; \text{V}_{\rho})$.

How can we get an action on $H'(F_n; V)$? What we need is for $b$ to act on $V$ so that the pullback and the action on $V$ cancel each other out.

This is exactly an action of $F_n \times B_n = \langle F, B | b x b^b = x^b \rangle$ on $V$.

Given such an action = a representation $F_n \times B_n \to \text{GL}(V)$ and $f \in H'(F_n; V)$,

define $b^* f \in H'(F_n; V)$ by $b^* f(x) = b \cdot f(x^b)$;

note that $b^* f(xy) = b \cdot f(x^b y^b) = b \cdot f(x^b) + b^* \cdot x^b \cdot f(y^b) = b \cdot f(x^b) + x \cdot b^* f(y) = b \cdot f(x) + x \cdot b^* f(y)$

so $b^* f$ really is a crossed homomorphism for the same action on $V$.

But where are we going to get representations of $F_n \times B_n$?
$F_n \times B_n < B_{n+1}$ — subgroup taking last point to itself

One last trick: “coloring” a representation by $t$. Let $t \in \mathbb{C}$ be fixed.

Given $\rho: F_n \times B_n \to GL(V)$, modify to get $\rho_t: F_n \times B_n \to GL(V)$
defined by $\rho_t(x_i) = t \cdot \rho(x_i)$, $\rho_t(b) = \rho(b)$.

Given $\rho: B_{n+1} \to GL(V)$, restrict to $F_n \times B_n$ and color by $t$; now we have an action of $B_n$ on $H'(F_n; V_\rho)$.

This is a new representation $\rho^t: B_n \to GL(H'(F_n; V_\rho))$ — and we can repeat!

Start with $\rho: B_{n+1} \to \mathbb{C}^*$ the trivial representation,

$\rho^t: B_n \to GL_n \mathbb{C}$ is the Burau representation (see below).

Start with $\rho: B_{n+1} \to GL_n \mathbb{C}$ the Burau representation,

$\rho^t: B_{n-1} \to GL_n \mathbb{C}$ is the Lawrence-Krammer representation $\rho_{LK}$.

Theorem (Bigelow, Krammer) $\rho_{LK}: B_{n-1} \to GL_n \mathbb{C}$ is injective.

Constructing the Burau representation

$F_n$: \[\bullet \quad \bullet \quad \bullet \quad \bullet \] Action of $B_n$: \[\bullet \quad \bullet \] $\sigma_i$ $\to$ $\sigma_i^{-1}$

$\rho_t: F_n \times B_n \to \mathbb{C}^*$

$x_i \to t$

$b \to 1$

$\mathbb{Z}'(F_n; C_t) = C_n$, with coordinates given by $f(x_i)$

$f \in \mathbb{Z}'(F_n; C_t) \Rightarrow f(x_i x_j) = f(x_i) + t^j f(x_i)$; note $f(x_i x_j) = f(x_i) + t^j f(x_i)$

$\Rightarrow f(x_i x_j) = -t f(x_i)$

Let’s compute action of $\sigma_i$ on $\mathbb{Z}'(F_n; C_t)$:

$\sigma_i f(x_i) = f(x_{i+1})$

$\sigma_i f(x_{i+1}) = f(x_{i+1} x_i x_{i+1})$

$= f(x_{i+1}) + t f(x_i) + t^2 f(x_{i+1})$

$= f(x_{i+1}) + t f(x_i) - t f(x_{i+1})$

In matrix form:

$\sigma_i \to \begin{pmatrix} 1 & 1 & t \\ 0 & 1 & 0 \end{pmatrix}$ and this is the standard definition of the Burau representation.