Cohomology of B_n

Recall $Y_n = (\mathbb{C}^n \setminus \bar{\Delta})/S_n$.

Viete map: think of $\mathbb{C}^n = (\lambda_1, \dots, \lambda_n)$, send $(\lambda_1, \dots, \lambda_n)$ to $t^n + z_1 t^{n-1} + \dots + z_n$, the monic polynomial with roots $\lambda_1, \dots, \lambda_n$

Since this doesn't depend on the order of the λ_i, descends to a map

$$\mathbb{C}^n/S_n \xrightarrow{\approx} \mathbb{C}^n$$
$$(\lambda_1, \dots, \lambda_n) \longmapsto (z_1, \dots, z_n)$$

explictly,
$$z_1 = \pm(\lambda_1 + \dots + \lambda_n)$$
$$z_2 = \pm(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \dots)$$
$$\vdots$$
$$z_n = \pm \lambda_1\lambda_2 \cdots \lambda_n$$

These are the elementary symmetric polynomials;

Fact: any symmetric polynomial in the λ_i is a polynomial in the z_i.

Algebro-geometrically:

ring of functions on $\mathbb{C}^n$ is $\mathbb{C}[\lambda_1, \dots, \lambda_n]$, and S_n acts on $\mathbb{C}[\lambda_1, \dots, \lambda_n]$

naive definition of $\mathbb{C}^n/S_n$ is as the variety whose ring of functions is the S_n-invariant functions $\mathbb{C}[\lambda_1, \dots, \lambda_n]^{S_n}$

Fact, restated: $\mathbb{C}[\lambda_1, \dots, \lambda_n]^{S_n} = \mathbb{C}[z_1, \dots, z_n]$. (this time, naive idea works)

$\bar{\Delta} \subset \mathbb{C}^n/S_n$ is sent to $D_0 \subset \mathbb{C}^n$, the space of polynomials with repeated roots.

D_0 is cut out by the polynomial $\prod_{i \neq j} (\lambda_i - \lambda_j)$;

since this is symmetric in the λ_i, it can be expressed as a polynomial in the z_i, called the discriminant

Examples: $n=2$: $(\lambda_1-\lambda_2)(\lambda_2-\lambda_1) = -\lambda_1^2 + 2\lambda_1\lambda_2 - \lambda_2^2 = -(\lambda_1+\lambda_2)^2 + 4\lambda_1\lambda_2 = -z_1^2 + 4z_2$ $\quad (b^2-4ac)$

$n=3$: $z_1^2 z_2^2 - 4z_2^3 - 4z_1^3 z_3 - 27z_3^2 - 18z_1z_2z_3$

So we need to understand the single hypersurface $D_0 \subset \mathbb{C}^n$.

Compactify to get $S^{2n} = \mathbb{C}^n \cup \{\infty\}$ and $D = D_0 \cup \{\infty\}$

Now Alexander duality gives:

$$H^*(B_n) = H^*(Y_n) = H^*(\mathbb{C}^n \setminus D_0) = H_{2n-*}(S^{2n}, D) = \tilde{H}_{2n-*+1}(D)$$

How do we analyze D and $\tilde{H}_*(D)$?

1) Set $D = D_n$ and consider all n together.
2) Filter D_n by $D_n = D_n^1 \supset D_n^2 \supset \dots$
   where D_n^k is the space of polynomials with at least k doubleroots.
   [$p(t) \in D_n^k \iff \exists\, q(t)$ w/ degree k s.t. $p(t) = (q(t))^2 r(t)$]
3) We have a natural multiplication map $D_n \times D_m \to D_{n+m}$
   $(p_1(t), p_2(t)) \mapsto p_1(t)p_2(t)$

   Since $D_n \times \{\infty\}$ and $\{\infty\} \times D_m$ are mapped to $\infty \in D_{n+m}$,
   this descends to a map $D_n \wedge D_m \to D_{n+m}$. (Recall $A \wedge B = A \times B / A \times \{*\} \cup \{*\} \times B$)
   Respects filtration: $D_n^k \wedge D_m^l \to D_{n+m}^{k+l}$.
4) Since each D_n^k is a manifold $\cup \{\infty\}$, can talk about degree of $D_n^k \wedge D_m^l \to D_{n+m}^{k+l}$.
5) "Filtrations" by n and k fit together to give something like a spectral sequence.

Theorems (Arnold):

1) $H^k(B_n) = 0$, $k \geq n$.
2) $H^*(B_n)$ is finite except for $H^0(B_n) = H^1(B_n) = \mathbb{Z}$
3) "stuttering": $H^*(B_{2n+1}) = H^*(B_{2n})$
4) "homological stability": $H^k(B_n) = H^k(B_{2k-2})$, $n \geq 2k-2$.

$H^*(B_\infty; \mathbb{Z}/p\mathbb{Z})$ known for all p (even as a Hopf algebra: put B_3 next to B_5, get B_8)

$H_*(B_\infty) \simeq H_*(\Omega^2_0 S^2)$: recall $\Omega^2 S^2 = \{\text{maps } S^2 \to S^2, * \to *\}$
components indexed by $\pi_0 \Omega^2 S^2 = \pi_2 S^2 = \mathbb{Z}$ (degree)

instead of X_n = space of n points in the plane, take $\mathcal{D}_n$ = space of disks in the plane
$X_n \simeq \mathcal{D}_n$ and we have a natural map $\mathcal{D}_n \to \Omega^2_n S^2$ by sending everything outside the disks to the basepoint and wrapping each disk around S^2 in a standard way.
Fixing an identification $\Omega^2_n S^2 \simeq \Omega^2_0 S^2$, the hard part is to show we get an isomorphism $H_k(\mathcal{D}_n) \simeq H_k(\Omega^2_0 S^2)$ as $n \to \infty$.

The spaces $\mathcal{D}_n$ fit together to give the little 2-disks operad, used by May and Boardman-Vogt in:

Recognition principle for double loop spaces: $X \simeq \Omega^2 Y \iff X$ is an algebra over the little 2-disks operad $\mathcal{D}$.

Representations of braid groups from cohomology

B_n = homeomorphisms of a disk fixing $\{n \text{ points}\}$ / isotopy fixing those points

So B_n "acts" on $D \setminus \{1, \ldots, n\}$ and acts on $\pi_1(D \setminus \{1, \ldots, n\}) = F_n$

we could look at action on $H^1(D \setminus \{1, \ldots, n\}) = H^1(F_n)$, but $H^1(F_n) = \mathbb{Z}^n$ and this is just the permutation representation $B_n \to S_n$

How about action on $H^1(F_n; V)$ for some nontrivial F_n-module V?

Not so fast: say $\rho: F_n \to GL(V)$ makes V an F_n-module;
then $H^1(F_n; V)$ is represented by crossed homomorphisms
$f: F_n \to V$ satisfying $f(xy) = f(x) + x \cdot f(y)$, where
x acts on $f(y) \in V$ by $\rho(x)$.

Now if $b: F_n \to F_n$, we want to pull back: $b^* f(x) = f(x^b)$

<u>BUT</u> this is no longer a crossed homomorphism for ρ; we have $b^* f \in H^1(F_n; V_{b^*\rho})$.

How can we get an action on $H^1(F_n; V)$? What we need is for b to act on V so that the pullback and the action on V cancel each other out.

This is exactly an action of $F_n \rtimes B_n = \langle F, B \mid b^{-1} x b = x^b \rangle$ on V.

Given such an action = a representation $F_n \rtimes B_n \to GL(V)$ and $f \in H^1(F_n; V)$,
define ${}^b f \in H^1(F_n; V)$ by ${}^b f(x) = b \cdot f(x^b)$;
note that ${}^b f(xy) = b \cdot f(x^b y^b) = b \cdot f(x^b) + b \cdot x^b \cdot f(y^b) = b \cdot f(x^b) + x \cdot b f(y^b) = {}^b f(x) + x \cdot {}^b f(y)$

So ${}^b f$ really is a crossed homomorphism for the <u>same</u> action on V.

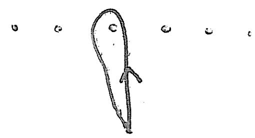
But where are we going to get representations of $F_n \rtimes B_n$?

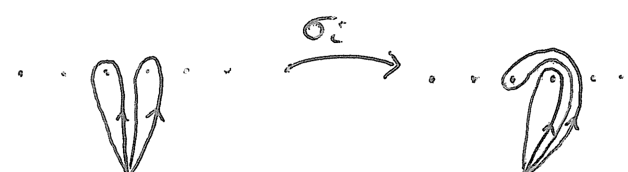
$F_n \rtimes B_n < B_{n+1}$ — subgroup taking last point to itself

One last trick: "coloring" a representation by t. Let $t \in \mathbb{C}$ be fixed.
Given $\rho : F_n \rtimes B_n \to GL(V)$, modify to get $\rho_t : F_n \rtimes B_n \to GL(V)$
defined by $\rho_t(x_i) = t \cdot \rho(x_i)$, $\rho_t(b) = \rho(b)$.

Given $\rho : B_{n+1} \to GL(V)$, restrict to $F_n \rtimes B_n$ and color by t;
now we have an action of B_n on $H^1(F_n; V_{\rho_t})$.
This is a new representation $\rho^+ : B_n \to GL(H^1(F_n; V_{\rho_t}))$ — and we can repeat!
Start with $\rho : B_{n+1} \to \mathbb{C}^\times$ the trivial representation,
$\rho^+ : B_n \to GL_n\mathbb{C}$ is the Burau representation (see below).
Start with $\rho : B_n \to GL_n\mathbb{C}$ the Burau representation,
$\rho^+ : B_{n-1} \to GL_{n^2}\mathbb{C}$ is the Lawrence-Krammer representation ρ_{LK}.
Theorem (Bigelow, Krammer): $\rho_{LK} : B_{n-1} \to GL_{n^2}\mathbb{C}$ is injective.

Constructing the Burau representation

F_n: 

Action of B_n: 

$x_i \longmapsto x_{i+1}$
$x_{i+1} \longmapsto x_{i+1} x_i x_{i+1}^{-1}$

$\rho_t : F_n \rtimes B_n \to \mathbb{C}^\times$
$x_i \longmapsto t$
$b \longmapsto 1$

$Z^1(F_n; \mathbb{C}_t) = \mathbb{C}^n$, with coordinates given by $f(x_i)$
$f \in Z^1(F_n; \mathbb{C}_t) \Rightarrow f(xy) = f(x) + t f(y)$; note $f(x_i x_i^{-1}) = f(x_i) + t f(x_i^{-1})$
$\Rightarrow f(x_i^{-1}) = -t^{-1} f(x_i)$

Let's compute action of σ_i on $Z^1(F_n; \mathbb{C}_t)$:

$\sigma_i f(x_i) = f(x_{i+1})$ $\quad \sigma_i f(x_{i+1}) = f(x_{i+1} x_i x_{i+1}^{-1})$ $\quad \sigma_i f(x_j) = f(x_j)$

$$= f(x_{i+1}) + t f(x_i) + t^2 f(x_{i+1}^{-1})$$
$$= f(x_{i+1}) + t f(x_i) - t f(x_{i+1})$$

In matrix form:

$$\sigma_i \longmapsto \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1-t & 1 & \\ & & t & 0 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

and this is the standard definition of the Burau representation.