Proof 2: our relation only involves 3 strands; should be supported on smaller subgroup.

Forgetting all but strands i,j,k gives map $P_n \to P_3$.

Since $W_{ij}$ pulls back to $W_{ij}$, suffices to show $R_{123} = 0$ in $H^2(P_3)$.

Let's understand $P_3$: generated by $A = T_{12}$, $B = T_{13}$, $C = T_{23}$.

\[
\begin{array}{c}
\text{turn your head by } \frac{2\pi}{3} \text{ and then by } \frac{4\pi}{3}, \text{ and same argument shows}
\end{array}
\]

\[ABC = BCA = CAB = \text{rotation through } 2\pi \text{ (also notice that } AB \text{ and } C \text{ commute).}\]

Lemma: $\langle ABC | ABC = BCA = CAB \rangle$.

Proof: Can derive from $P_3 = F_2 \times Z = \langle A, C \rangle \times \langle B \rangle$.

Idea of proof: pretend $P_3 = F_3$, then show dreams come true.

Let $x, y, z: P_3 \to Z$ be dual to $A, B, C$. Pull back to $F_3 = \langle A, B, C \rangle \to P_3$.

Certainly $x + y + z = 0$ in $H^2(F_3)$ b/c $H^2(F_3) = 0$, so there is some cochain $\gamma \in C^1(F_3; Z)$ such that $\delta \gamma = f(x) + f(y) + f(z)$.

We will show that $\delta \gamma$ descends to $C'(P_3; Z)$ so that $R_{123} = \delta \gamma = 0$ in $H^2(P_3)$.

Simpler example: $x + y \in C^2(F_3; Z)$ maps $(g_1, g_2) \mapsto \alpha(g_1) \beta(g_2)$.

We want $f: F_3 \to Z$ so that $f(g_1, g_2) = f(g_1) + f(g_2) + (\alpha(g_1) \beta(g_2)) \odot [\delta f = x + y]$.

Can normalize so that $f(0, 0) = 0$ by adding elements of $Z'(F_3; Z) = \text{Hom}(F_3, Z)$.

Then $f$ implies $f(AB) = 0$, $f(AB) = 1$, $f(AB) = 1$, $f(ABAB) = 3$; we see that $f$ counts how many $A$s are to the left of $B$s, how many Bs.

This formula does not give a well-defined map in $C'(P_3; Z)$, because we would have $f(ABC) = 1 \neq 0 = f(BCA)$ (which is good, because $x + y + z = 0$).

But $\delta \gamma$, which counts $A$s to the left of $B$s, plus $B$s to the left of $C$s, plus $C$s to the left of $A$s, is well-defined.

Key observation: $\delta \gamma = x + y + z \in C^2(P_3; Z)$. (Easier to see $\delta f = x + y$.)

Thus $R_{123} = \delta \gamma$ and thus is 0 in $H^2(P_3)$.
Proof 3: Commutators and cup product

[ , ] : G × G → [G, G] set map \[ [x, y^z] = [x, y] y [x, z] y^{-1} \]
\[(x, y) \mapsto [x, y] \]

Since \([G, G] \times G, G \times G, G]\) map to \([\mathcal{H}, \mathcal{H}, G]\), descends to bilinear map

\[ [ , ] : H_1(G) \wedge H_1(G) \rightarrow [G, G]/[[G, G], G] \]
always onto

Theorem (Sullivan, Chen)

\[ 0 \rightarrow \left( \frac{[G, G]}{[[G, G], G]} \right) \xrightarrow{\sim} H^1(G) \wedge H^1(G) \xrightarrow{\wedge} H^2(G) \text{ is exact.} \]

Examples:

- \(G = \mathbb{Z}^2\)

\[ 0 \rightarrow 0 \rightarrow \mathbb{Z}^2 \xrightarrow{\sim} \mathbb{Z} \]

- \(G = \pi_1(S_3) = \langle a_i, b_i \mid [a_i, b_i] \ldots [a_g, b_g] = 1 \rangle \quad H_1(G) = \langle A_i, B_i \rangle \quad H^1(G) = \langle \alpha_i, \beta_i \rangle \)

\[ 0 \rightarrow \Lambda^2 H^1(G) \xrightarrow{\cdot , \cdot} \Lambda^2 H^1(G) \xrightarrow{\wedge} \mathbb{Z} \quad \text{\(\omega\) symplectic form} \]

\[ \text{im} \ [ , \ ] = \{ \lambda \in \Lambda^2 H^1(G) \mid \lambda(A_i \wedge B_i \ldots A_g \wedge B_g) = 0 \} \subset \ker \omega \]

- Heisenberg group \(N = \langle x, y, z \mid [x, y] = z, [x, z] = [y, z] = 1 \rangle \)

\[ 0 \rightarrow \mathbb{Z} \xrightarrow{\sim} \Lambda^2 \mathbb{Z}^2 \rightarrow \mathbb{Z}^2 \quad \text{thus the cup product} \]

\[ H^1(N) \wedge H^1(N) \rightarrow H^2(N) \text{ is trivial} \]

pure braid group \(P_3 = \langle a, b, c \mid abc = bca = caba \rangle \quad H_1(P_3) = \langle AB, C \rangle \)

Note \([ab, c] = 1 \]
\([bc, a] = 1 \]
\([ca, b] = 1 \]

and thus \([P_3, P_3]/[P_3, P_3], P_3 = \Lambda^2 H^1(P_3) \)

\[ \langle (A + B) \wedge C, (B + C) \wedge A, (C + A) \wedge B \rangle. \]

It follows that \(\text{im} \ [ , \ ] = \{ \lambda \in \Lambda^2 H^1(P_3) \mid \lambda(A + B) \wedge C = 0, \lambda(B + C) \wedge A = 0, \lambda((C + A) \wedge B) = 0 \} \)

\[ = \{ \lambda \in \Lambda^2 H^1(P_3) \}
\[ \lambda(A \wedge B) = \lambda(B \wedge C) = \lambda((C \wedge A) \}

\[ = \langle \alpha \wedge \beta + \beta \wedge \gamma + \gamma \wedge \alpha \rangle, \]

and the kernel of \(H^1(P_3) \wedge H^1(P_3) \rightarrow H^2(P_3)\) is spanned by \(\alpha \wedge \beta + \beta \wedge \gamma + \gamma \wedge \alpha = R_{32} \). \(\blacksquare\)
Stein spaces

We calculated $H^*(\mathbb{P}^n)$ and found that $H^i(\mathbb{P}^n) = 0$, i.e.,

we could have done almost as well without calculation:

A **Stein space** is a complex manifold satisfying certain vanishing theorems. A good heuristic is that an open domain $\Omega \subset \mathbb{C}^n$ is Stein if for every point $p \in \Omega$, there is a function holomorphic on $\Omega$ and blowing up at $p$.

Examples:

- Any domain in $\mathbb{C}$ is Stein.
- The ball $B_r = \{ z_i \in \mathbb{C}^n \mid \sum |z_i|^2 < 1 \}$ is Stein.
- The configuration space $X_n$ (indeed any hypersurface complement) is Stein.

**Proof**: Use the functions $\frac{z_1}{z_2 \cdots z_n}$, holomorphic on $X_n$.

Non-examples:

- The annulus $B_2 \setminus \overline{B}_1$ is not Stein.
- The punctured space $\mathbb{C}^n \setminus \{0\} \setminus \overline{B_1}$ is not Stein.

**Theorem**: A Stein space of complex dimension $n$ is homotopy equivalent to a CW complex of dimension $n$.

**Corollary**: $\mathbb{P}^n$ has a $K(\mathbb{P}^n, 1)$ CW complex of dimension $n$.

(Explicit complexes known, e.g. Salvetti complex associated to any hyperplane arrangement)

Proof: relies on fact that $X$ is biholomorphic to a complex submanifold closed in $\mathbb{C}^n$.

For example, $X_n \hookrightarrow \mathbb{C}^n$, $N = n + 1$

$$(z_1) \mapsto (z_1, \frac{1}{z_1 - z_2}, \ldots)$$

Corresponds to $\mathbb{R}\setminus \mathbb{S}^{n-1} \rightarrow \mathbb{R}^2$

$$x \mapsto (x, \frac{1}{x})$$
Morse theory and focal points

Let $M^k \subset \mathbb{R}^n$ be a real submanifold closed in $\mathbb{R}^n$.

Natural map from normal bundle $NM^k \to \mathbb{R}^n$.

A focal point is a critical value of this map.

(where normals focus). Rank of a focal point is nullity of Jacobian there.

By Sard's theorem, focal points have measure 0.

Lemma: For $q \in M^k$ and $\nu$ normal to $T^k$ at $q$, let $K_1, \ldots, K_n$ be the principal curvatures of $M$ at $q$ in the direction of $\nu$.

Then the focal points on the line $q + t\nu$ occur at $q + K_i^{-1}\nu$ (with multiplicity).

Corollary: there are at most $n$ focal points (with multiplicity) on such a line.

Theorem: For $p \in \mathbb{R}^n$, define $L_p : \mathbb{R}^n \to \mathbb{R}$ by $L_p(q) = ||p - q||^2$. Restrict to $M^k$.

Then

1) $q$ is a critical point of $L_p \iff$ the line from $q$ to $p$ is normal to $M$.
2) $q$ is a degenerate critical point $\iff p$ is a focal point (Corollary: $L_p$ is Morse for almost all $p$).
3) the index of $q$ is the number of focal points (with multiplicity) on the line from $q$ to $p$.

Lemma: Let $M^{2k} \subset \mathbb{R}^{2n}$ as above be a complex submanifold.

Then the focal points on line $q + t\bar{\nu}$ are symmetrically distributed; if $q + t\bar{\nu}$ has multiplicity $m$, $q - t\bar{\nu}$ is a focal point with same multiplicity.

Corollary: in this case, all critical points have index $\leq k$.

Proof: on each line $q + t\bar{\nu}$ at most $2k$ focal points occur; by symmetry at most $k$ are on one side of $M$, and thus at most $k$ occur on the interval from $q$ to $p$.

Corollary: A complex submanifold $M^{2k} \subset \mathbb{R}^{2n}$ closed in $\mathbb{R}^{2n}$ has the homotopy type of a CW complex of dimension $\leq k$. 