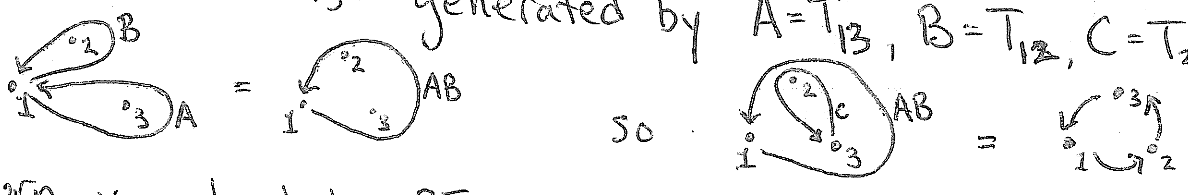
Proof 2: our relation only involves 3 strands; should be "supported on smaller subgroup".

Forgetting all but strands i,j,k gives map $P_n \to P_3$;
since W_{ij} pulls back to W_{ij}, suffices to show $R_{123}=0$ in $H^2(P_3)$.

Let's understand P_3: generated by $A=T_{13}$, $B=T_{12}$, $C=T_{23}$.



rotation through 2π

turn your head by $\frac{2\pi}{3}$ and then by $\frac{4\pi}{3}$, and same argument shows
$ABC = BCA = CAB$ = rotation through 2π (also notice that AB and C commute).

Lemma: $P_3 = \langle A,B,C \mid ABC = BCA = CAB \rangle$.

Proof: Can derive from $P_3 = F_2 \rtimes \mathbb{Z} = \langle A, C\rangle \rtimes \langle B \rangle$.

Idea of proof: pretend $P_3 = F_3$, then show dreams come true.

Let $\alpha,\beta,\gamma : P_3 \to \mathbb{Z}$ be dual to A,B,C. Pull back to $F_3 = \langle A,B,C\rangle \to P_3$.

Certainly $\alpha\smile\beta + \beta\smile\gamma + \gamma\smile\alpha = 0$ in $H^2(F_3)$ b/c $H^2(F_3)=0$, so there is some cochain
$\varphi \in C^1(F_3;\mathbb{Z})$ = Set-maps $F_3 \to \mathbb{Z}$ s.t. $\delta\varphi = \alpha\smile\beta+\beta\smile\gamma+\gamma\smile\alpha$.

We will show that φ descends to $C^1(P_3;\mathbb{Z})$ so that $R_{123} = \delta\varphi = 0$ in $H^2(P_3)$.

<u>Simpler example</u>: $\alpha\smile\beta \in C^2(F_3;\mathbb{Z})$ maps $(g_1,g_2) \longmapsto \alpha(g_1)\cdot\beta(g_2)$.

We want $f : F_3 \to \mathbb{Z}$ so that $f(g_1 g_2) = f(g_1) + f(g_2) + \alpha(g_1)\beta(g_2)$ (*) $[\delta f = \alpha\smile\beta]$.

Can normalize so that $f(A)=f(B)=f(C)=0$ by adding elements of $Z^1(F_3;\mathbb{Z}) = \mathrm{Hom}(F_3;\mathbb{Z})$

Then (*) implies $f(BA)=0$, $f(AB)=1$, $f(BAB)=1$, $f(ABAB)=3$; we see that f counts how many As are to the left of how many Bs.

For any $g = g_1\cdots g_k$, $f(g) = \sum_{i<j} \alpha(g_i)\beta(g_j)$; if well-defined, this determines f uniquely.

This formula does not give a well-defined map in $C^1(P_3;\mathbb{Z})$,
because we would have $f(ABC) = 1 \neq 0 = f(BCA)$. (which is good, because $\alpha\smile\beta \neq 0$ in $H^2(P_3)$).

But φ, which counts As to the left of Bs, plus Bs to the left of Cs, plus Cs to the left of As, is well-defined,
because replacing ABC with BCA or CAB leaves φ constant. (Exercise.)

Key observation: $\delta\varphi = \alpha\smile\beta + \beta\smile\gamma + \gamma\smile\alpha \in C^2(P_3;\mathbb{Z})$. (Easier to see $\delta f = \alpha\smile\beta$.)

Thus $R_{123} = \delta\varphi$ and thus is 0 in $H^2(P_3)$.

Proof 3:
Commutators and cup product ⑤

$$[\ ,\ ] : G\times G \to [G,G] \qquad \text{set map} \qquad [x,yz]=[x,y]\,y[x,z]y^{-1}$$
$$(x,y)\longmapsto [x,y]$$

since $[G,G]\times G$, $G\times[G,G]$ map to $[[G,G],G]$, descends to bilinear map

$$[\ ,\ ]: H_1(G)\wedge H_1(G) \longrightarrow [G,G]/[[G,G],G] \qquad \text{always onto}$$

Theorem (Sullivan, Chen)

$$0\to\left([G,G]/[[G,G],G]\right)^* \xrightarrow{[\ ,\ ]^*} H^1(G)\wedge H^1(G) \xrightarrow{\wedge} H^2(G) \quad \text{is exact.}$$

Examples:

- $G=\mathbb{Z}^2$

$$0\longrightarrow 0 \longrightarrow \Lambda^2\mathbb{Z}^2 \xrightarrow{\approx} \mathbb{Z}$$

- $G=\pi_1(\Sigma_g)=\langle a_i,b_i \mid [a_1,b_1]\cdots[a_g,b_g]=1\rangle \qquad H_1(G)=\langle A_i,B_i\rangle \qquad H^1(G)=\langle \alpha_i,\beta_i\rangle$

$$0\to\left(\Lambda^2H_1(G)/A_1\wedge B_1+\cdots+A_g\wedge B_g\right)\xrightarrow{[\ ,\ ]^*}\Lambda^2H^1(G)\xrightarrow{\omega}\mathbb{Z}$$

ω symplectic form
$\alpha_i\wedge\beta_i \longmapsto 1$
$\left.\begin{matrix}\alpha_i\wedge\alpha_j\\ \beta_i\wedge\beta_j\\ \alpha_i\wedge\beta_j\end{matrix}\right\}\longmapsto 0$

$$\operatorname{im}[\ ,\ ]^*=\{\lambda\in\Lambda^2H^1(G) \mid \lambda(A_1\wedge B_1+\cdots+A_g\wedge B_g)=0\}=\ker\omega$$

- Heisenberg group $N=\langle x,y,z \mid [x,y]=z,\ [x,z]=[y,z]=1\rangle$

$$0\to\mathbb{Z}\xrightarrow{\approx}\Lambda^2\mathbb{Z}^2\longrightarrow\mathbb{Z}^2$$

thus the cup product $H^1(N)\wedge H^1(N)\longrightarrow H^2(N)$ is trivial

- pure braid group $P_3=\langle a,b,c \mid abc=bca=cab\rangle \quad H_1(P_3)=\langle A,B,C\rangle$ Note $[ab,c]=1$, $[bc,a]=1$, $[ca,b]=1$,

and thus
$$[P_3,P_3]/[[P_3,P_3],P_3]=\Lambda^2H_1(P_3)/\langle (A+B)\wedge C,\ (B+C)\wedge A,\ (C+A)\wedge B\rangle$$

It follows that
$$\begin{aligned}\operatorname{im}[\ ,\ ]^* &=\{\lambda\in\Lambda^2H^1(P_3) \mid \lambda((A+B)\wedge C)=0,\ \lambda((B+C)\wedge A)=0,\ \lambda((C+A)\wedge B)=0\}\\ &=\{\lambda\in\Lambda^2H^1(P_3) \mid \lambda(A\wedge B)=\lambda(B\wedge C)=\lambda(C\wedge A)\}\\ &=\langle \alpha\wedge\beta+\beta\wedge\gamma+\gamma\wedge\alpha\rangle,\end{aligned}$$

and the kernel of $H^1(P_3)\wedge H^1(P_3)\longrightarrow H^2(P_3)$ is spanned by $\alpha\wedge\beta+\beta\wedge\gamma+\gamma\wedge\alpha=R_{123}$. ▨

Stein spaces

We calculated $H^\bullet(P_n)$ and found that $H^i(P_n)=0$, $i \geq n$.
We could have done almost as well without calculation:

A <u>Stein space</u> is a complex manifold satisfying certain vanishing theorems
a good heuristic is that an open domain $\Omega \subset \mathbb{C}^n$
is Stein if for every point $p \in \partial\Omega$, there is a function holomorphic on Ω and blowing up at p.

Examples:

Any domain in $\mathbb{C}$ is Stein.
The ball $B_r = \{(z_i) \in \mathbb{C}^n \mid \sum \|z_i\|^2 < 1\}$ is Stein.
The configuration space X_n (indeed any hypersurface complement) is Stein.
Pf: Use the functions $\frac{1}{z_i - z_j}$, holomorphic on X_n.

Non-examples:

The annulus $B_2 \setminus \overline{B_1}$ is not Stein.
The punctured space $\mathbb{C}^n \setminus \{0\}$ or $B_r \setminus \{0\}$ is not Stein.

Theorem: A Stein space X of complex dimension n is homotopy equivalent to a CW complex of dimension n.

Corollary: P_n has a $K(P_n, 1)$ CW complex of dimension n.
(Explicit complexes known, e.g. Salvetti complex associated to any hyperplane arrangement)

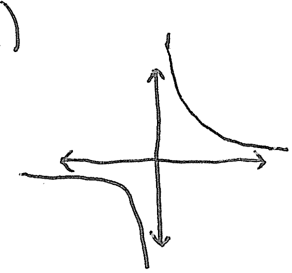
Proof: relies on fact that X is biholomorphic to a complex submanifold closed in $\mathbb{C}^N$.

For example, $X_n \hookrightarrow \mathbb{C}^N$, $N = n + \binom{n}{2}$

$(z_i) \longmapsto (z_i \ldots, \frac{1}{z_i - z_j} \ldots)$

Corresponds to $\mathbb{R} \setminus \{0\} \hookrightarrow \mathbb{R}^2$

$x \longmapsto (x, \frac{1}{x})$



Morse theory and focal points

Let $M^k \subset \mathbb{R}^N$ be a real submanifold closed in $\mathbb{R}^N$.

Natural map from normal bundle $NM^k \to \mathbb{R}^N$.

A <u>focal point</u> is a critical value of this map.
(where normals focus). Rank of a focal point is nullity of Jacobian there.

By Sard's theorem, focal points have measure 0.

Lemma: For $q \in M^k$ and $\vec{v}$ normal to M^k at q, let $K_1, \dots, K_n$ be the principal ~~radii of~~ curvatures of M at q in the direction of $\vec{v}$.
Then the focal points on the line $q + t\vec{v}$ occur at $q + K_i^{-1}\vec{v}$ (with multiplicity).

Corollary: there are at most n focal points (with multiplicity) on such a line.

Theorem: For $p \in \mathbb{R}^N$, define $L_p : \mathbb{R}^N \to \mathbb{R}$ by $L_p(q) = \|p - q\|^2$. Restrict to M^k.
Then
1) $\vec{q}$ is a critical point of $L_p \Leftrightarrow$ the line from q to p is normal to M.
2) q is a degenerate critical point $\Leftrightarrow p$ is a focal point (Corollary: L_p is Morse for almost all p)
3) the index of q is the number of focal points (with multiplicity) on the line from q to p.

Lemma: Let $M^{2k} \subset \mathbb{R}^{2N}$ as above be a <u>complex</u> submanifold.
Then the focal points on line $q + t\vec{v}$ are symmetrically distributed; if $q + t\vec{v}$ has multiplicity m, $q - t\vec{v}$ is a focal point with same multiplicity.

Corollary: in this case, all critical points have index $\leq k$.

Proof: on each line $q + t\vec{v}$ at most $2k$ focal points occur; by symmetry at most k are on one side of M, and thus at most k occur on the interval from q to p.

Corollary: A complex submanifold $M^{2k} \subset \mathbb{R}^{2N}$ closed in $\mathbb{R}^{2N}$ has the homotopy type of a CW complex of dimension $\leq k$.