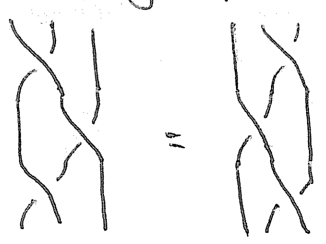
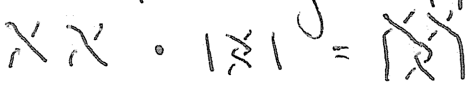
Definitions, background

Braid group: $B_n :=$ n strands montonically embedded in $\mathbb{C} \times I$
$= \{1, 2, \ldots, n\}$ on top + bottom
modulo time-preserving isotopy



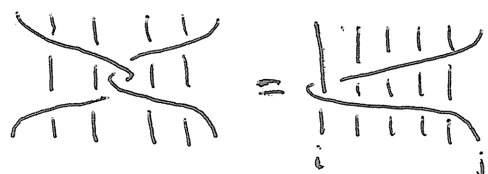
ABA = BAB
w/ $A = \sigma_i$, $B = \sigma_{i+1}$
"braid relation"

Composition: 
Identity: | | | |
Generators:  σ_i Proof: avoid triple points

Theorem (Artin): $B_n = \left\langle \sigma_i \;\middle|\; \begin{array}{l} \sigma_i \text{ and } \sigma_{i+1} \text{ braid} \\ \sigma_i \text{ and } \sigma_j \text{ commute, } |i-j|>1 \end{array} \right\rangle$

By following strands: $1 \to P_n \to B_n \to S_n \to 1$
$\sigma_i \longmapsto (i\ i{+}1)$

P_n "pure braid group" generated by



$T_{i,j}$

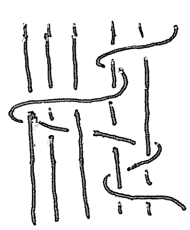
Proof 1: S_n has presentation $\left\langle s_i = (i\ i{+}1) \;\middle|\; \begin{array}{l} s_i^2 = 1 \\ s_i \text{ and } s_{i+1} \text{ braid} \\ s_i \text{ and } s_j \text{ commute} \end{array} \right\rangle$ (Coxeter group)

General lemma:
Given $1 \to K \to G \xrightarrow{\pi} H \to 1$, g_i generators for G, and a presentation of H in terms of $\pi(g_i)$
K is normally generated in G by those relations R_j that <u>don't</u> lift to G.
Proof: $G/\langle R_j \rangle$ is a presentation for H.

So lifting s_i to σ_i, we find P_n is generated by the B_n-conjugates of $\sigma_i^2 = T_{i,i+1}$.
Check that $T_{i,j}$ generate all conjugates. ∎

Proof 2. Forgetting last strand gives

kernel:



$1 \to F_n \to P_{n+1} \to P_n \to 1$

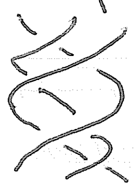
$P_{n+1} = F_n \rtimes P_n$

obvious section $P_n \to P_{n+1}$ (add strand)

By induction, P_n generated by $T_{i,j}$
F_n generated by $T_{i,n+1}$. ∎

$P_{n+1} = F_n \rtimes P_n$ yields presentation for P_n inductively.

Configuration spaces:

Consider  as a movie $\begin{matrix} t=0 \\ \downarrow \\ t=1 \end{matrix}$, we see n points moving in the plane.

Each strand traces out $\gamma_i(t)$, w/ $\gamma_i(t) \neq \gamma_j(t)$ and $\gamma_i(0)=\gamma_i(1)=i$ (pure braid).

Key definition: $X_n := \mathbb{C}^n \setminus \bar{\Delta} = \mathbb{C}^n \setminus \{\text{any } z_i = z_j\}$

$\bar{p} = (1, \dots, n)$

a pure braid $\rightsquigarrow$ an n-strand movie $\rightsquigarrow$ n loops γ_i $\rightsquigarrow$ one loop $\bar{\gamma}$ in X_n

$\bar{\gamma}(0)=\bar{\gamma}(1)=\bar{p}$ level-preserving isotopy $\longleftrightarrow$ homotopy of $\bar{\gamma}$

Thus: $P_n = \pi_1(X_n, \bar{p})$

If $Y_n = X_n/S_n$ is the unordered configuration space, $B_n = \pi_1(Y_n, \bar{p})$

Cohomology groups of P_n

Natural projection $X_{n+1} \to X_n$, fiber over $\bar{p}=(1,\dots,n)$ is $\mathbb{C}\setminus\{1,\dots,n\}$, is a fiber bundle.

Corollary: X_n is a $K(P_n,1)$. Proof: LES in homotopy shows $F \to E \to B$ if F and B are aspherical, so is E. Induction on n.

Claim: $H^*(P_n) \approx H^*(\mathbb{Z}) \otimes H^*(F_2) \otimes H^*(F_3) \otimes \cdots \otimes H^*(F_{n-1})$ <u>as abelian groups.</u>

$= H^*(\mathbb{Z} \times F_2 \times F_3 \times \cdots \times F_{n-1})$

Proof: Consider Leray-Serre spectral sequence for $\mathbb{C}\setminus\{1,\dots,n\} \to X_{n+1} \to X_n$.

Note we have a section (explicitly $(z_i) \longmapsto (z_i, \frac{z_1+\cdots+z_n}{n} + 2\cdot\max|z_i - z_j| + 1)$) corresponding to splitting $P_{n+1} = F_n \rtimes P_n$.

Since $H^*(\mathbb{C}\setminus\{1,\dots,n\})$ is only $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^n$, and action of P_n is trivial (pure braid group),

E_2 page looks like

$(E_2^{p,q} = H^p(B; H^q(F)))$

$$\begin{matrix} H^0(X_n)\otimes\mathbb{Z}^n & H^1(X_n)\otimes\mathbb{Z}^n & H^2(X_n)\otimes\mathbb{Z}^n & \cdots & H^n(X_n)\otimes\mathbb{Z}^n \\ H^0(X_n) & H^1(X_n) & H^2(X_n) & \cdots & H^n(X_n) \end{matrix}$$

(same as for direct product $F_n \times P_n$)

Existence of section $X_n \to X_{n+1}$ implies d_2 is zero (black box), and all later differentials are 0, so $E_2 = E_3 = \cdots = E_\infty$. By induction, everything is free abelian, so no extension problems. ■

Corollary: $H^*(P_n)$ is torsion-free.

Cup product in $H^*(P_n)$

Generators for $H^1(P_n) \simeq \mathrm{Hom}(P_n, \mathbb{Z})$ given by

$W_{ij} : P_n \to \mathbb{Z}$ measuring winding of i^{th} strand around j^{th}

$W_{ij}(T_{ij}) = 1$, $W_{ij}(T_{..}) = 0$

Fundamental relation:

$$W_{ij} \wedge W_{jk} + W_{jk} \wedge W_{ki} + W_{ki} \wedge W_{ij} = 0 \quad \text{in } H^2(P_n) \qquad (\text{call this } R_{ijk} = 0)$$

Corollary: $H^*(P_n) = \Lambda^*[W_{ij}] / (R_{ijk})$.

Proof: spectral sequence implies $H^*(P_n)$ generated by W_{ij}, so the natural map $\Lambda^*[W_{ij}]/(R_{ijk}) \to H^*(P_n)$ is onto. Calculation shows ranks are the same, thus isomorphism.

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3 proofs of fundamental relation (de Rham, bar cochain, Sullivan theory)

Proof 1:

Define $\omega_{ij} = \dfrac{dz_i - dz_j}{z_i - z_j} \in \Omega^1(X_n; \mathbb{C})$.

Note $\omega_{ij} = d\left(\log(z_i - z_j)\right)$ so ω_{ij} is closed;

Cauchy integral formula $\Rightarrow \oint_\gamma \omega_{ij}$ = winding number of $\gamma_i(t)$ around $\gamma_j(t)$

so ω_{ij} represents $W_{ij} \in H^1(X_n; \mathbb{C})$.

Claim: the 2-form below is identically 0 at every point:

$$\rho_{ijk} = \left(\frac{dz_i - dz_j}{z_i - z_j}\right) \wedge \left(\frac{dz_j - dz_k}{z_j - z_k}\right) + \left(\frac{dz_j - dz_k}{z_j - z_k}\right) \wedge \left(\frac{dz_k - dz_i}{z_k - z_i}\right) + \left(\frac{dz_k - dz_i}{z_k - z_i}\right) \wedge \left(\frac{dz_i - dz_j}{z_i - z_j}\right)$$

Proof of claim: would suffice to expand and collect terms.

We can set $dz_i - dz_j = A$, $z_i - z_j = a$, etc., so this is $\dfrac{A \wedge B}{ab} + \dfrac{B \wedge C}{bc} + \dfrac{C \wedge A}{ca}$.

Clear denominators and note $A + B + C = 0$, $a + b + c = 0$.

$$\begin{aligned} abc\,\rho_{ijk} &= c\,A \wedge B + a\,B \wedge C + b\,C \wedge A \\ &= (-a-b) A \wedge B + a B \wedge (-A - B) + b(-A - B) \wedge A \\ &= (-a - b + a + b) A \wedge B \\ &= 0. \quad \blacksquare \end{aligned}$$

Thus $R_{ijk} = 0$ in $H^2(P_n; \mathbb{C})$. Since $H^*(P_n)$ is torsion-free, $R_{ijk} = 0$ in $H^2(P_n)$. $\blacksquare$

Corollary to claim: $W_{ij} \mapsto \omega_{ij}$ defines an injection

$$H^*(P_n) \hookrightarrow \Omega^*(X_n; \mathbb{C})$$ into the algebra of closed differential forms.

Corollary: X_n is formal (in the sense of Sullivan theory).