Definitions, background

Braid group: \( B_n := \{1, 2, \ldots, n\} \) on top + bottom modulo time-preserving isotopy

Composition: \( \chi \chi' \cdot 1 \chi'' = \chi \chi' \chi'' \)

Identity: \( 1 \)

Generators: \( \sigma_i \)

Proof: avoid triple points

A = \sigma_i, B = \sigma_j

"braided relation" Theorem (Artin): \( B_n = \langle \sigma_i | \sigma_i \text{ and } \sigma_j \text{ braid, commute, } i \neq j \rangle \)

By following strands: \( 1 \to P_n \to B_n \to S_n \to 1 \)

\( \sigma_i \mapsto (i \ i+1) \)

P_n "pure braid group" generated by \( T_{i,j} \)

Proof 1: \( S_n \) has presentation

\[
\langle S_i = (i \ i+1) | S_i^2 = 1, S_i \text{ and } S_j \text{ braid, commute} \rangle
\]

(coxeter group)

General lemma:

Given \( 1 \to K \to G \to H \to 1 \), \( g_i \) generators for \( G \), and a presentation of \( K \) is normally generated in \( G \) by those relations \( R_j \) that don't lift to \( H \).

Proof: \( G/\langle R_j \rangle \) is a presentation for \( H \).

So lifting \( s_i \) to \( \sigma_i \), we find \( P_n \) is generated by the \( B_n \)-conjugates of \( \sigma_i^2 = T_{i,i+1} \).

Check that \( T_{i,j} \) generate all conjugates.

Proof 2. Forgetting last strand gives

\[
\begin{array}{c}
\text{kernel:} \\
\end{array}
\]

\( 1 \to F_n \to P_{n+1} \to P_n \to 1 \)

\( P_{n+1} = F_n \times P_n \)

By induction, \( P_n \) generated by \( T_{i,i+1} \)

\( F_n \) generated by \( T_{i+1,i} \).

\( P_{n+1} = F_n \times P_n \) yields presentation for \( P_n \) inductively.
Configuration spaces:
Consider \( \mathcal{S} \) as a movie \( I \), we see \( n \) points moving in the plane.

Each strand traces out \( \gamma_i(t) \), \( w/ \gamma_i(t) \neq \gamma_j(t) \) and \( \gamma_i(0)=\gamma_i(1)=i \) (pure braid).

Key definition: \( X_n := C^n \setminus \Delta = C^n \setminus \{ \text{any } z_i=z_j \} \)
\( \bar{p}=(1,\cdots,n) \) a pure braid \( \rightarrow \) an \( n \)-strand movie \( \rightarrow \) \( n \) loops \( \bar{\gamma} \) one loop \( \bar{\gamma} \) in \( X_n \)
\( \bar{\gamma}(0)=\bar{\gamma}(1)=\bar{p} \) level-preserving isotopy \( \rightarrow \) homotopy of \( \bar{\gamma} \)

Thus: \( P_n=\pi_1(X_n,\bar{p}) \)
If \( Y_n=X_n/S_n \) is the unordered configuration space, \( B_n=\pi_1(Y_n,\bar{p}) \)

Cohomology groups of \( P_n \)
Natural projection \( X_{n+1} \rightarrow X_n \), fiber over \( \bar{p}=(1,\cdots,n) \) is \( C \setminus \{ 1,\cdots,n \} \), is a fiber bundle.
Corollary: \( X_n \) is a \( K(P_n,1) \). Proof: LES in homotopy shows \( F \rightarrow E \rightarrow B \) if \( F \) and \( B \) are aspherical, so is \( E \). Induction on \( n \).

Claim: \( H^*(P_n) \cong H^*(Z) \otimes H^*(F_2) \otimes H^*(F_3) \otimes \cdots \otimes H^*(F_{n-1}) \) as abelian groups.

Proof: Consider Leray-Serre spectral sequence for \( C \setminus \{ 1,\cdots,n \} \rightarrow X_{n+1} \rightarrow X_n \).
Note we have a section (explicitly \( (z_i) \mapsto (z_i, \frac{z_1+\cdots+z_n}{n}+2 \max |z_i-z_j|+1) \))
corresponding to splitting \( P_n=F_n \times P_n \).

Since \( H^*(C \setminus \{ 1,\cdots,n \}) \) is only \( H^0=\mathbb{Z} \), \( H^1=\mathbb{Z}^n \), and action of \( P_n \) is trivial (pure braid group),

\( E_2 \) page looks like
\[
\begin{array}{c}
H^0(X_n) \otimes \mathbb{Z} & H^1(X_n) \otimes \mathbb{Z} & \cdots & H^n(X_n) \otimes \mathbb{Z} \\
H^0(F_n) \otimes \mathbb{Z} & H^1(F_n) \otimes \mathbb{Z} & \cdots & H^n(F_n) \otimes \mathbb{Z}
\end{array}
\]
(same as for direct product \( F_n \times P_n \))

Existence of section \( X_n \rightarrow X_{n+1} \) implies \( d_2 \) is zero (black box), and all later differentials are 0,
so \( E_2=E_3=\cdots=E_\infty \). By induction, everything is free abelian, so no extension problems.

Corollary: \( H^*(P_n) \) is torsion-free.
Cup product in $H^*(\mathbb{P}^n)$

Generators for $H^*(\mathbb{P}^n) \cong \text{Hom}(\mathbb{P}^n, \mathbb{Z})$ given by

$W_{ij}: \mathbb{P}^n \to \mathbb{Z}$ measuring winding of $i^{th}$ strand around $j^{th}$

$W_{ij}(T_{ij}) = 1$, $W_{ij}(T_{ji}) = 0$

Fundamental relation:

$W_{ij} \wedge W_{jk} + W_{jk} \wedge W_{ki} + W_{ki} \wedge W_{ij} = 0$ in $H^2(\mathbb{P}^n)$ (call this $R_{ijk} = 0$)

Corollary: $H^*(\mathbb{P}^n) = \Lambda^*[W_{ij}] / (R_{ijk})$.  
Proof: spectral sequence implies $H^*(\mathbb{P}^n)$ generated by $W_{ij}$, so the natural map $\Lambda^*[W_{ij}] / (R_{ijk}) \to H^*(\mathbb{P}^n)$ is onto. Calculation shows ranks are the same, thus isomorphism.

3 proofs of fundamental relation (de Rham, bar cochain, Sullivan theory)

Proof 1:
Define $\omega_{ij} = \frac{dz_i - dz_j}{z_i - z_j} \in \Omega^2(\mathbb{P}^n, \mathbb{C})$.

Note $\omega_{ij} = d(\log(z_i - z_j))$ so $\omega_{ij}$ is closed;

Cauchy integral formula $\Rightarrow \int_\gamma \omega_{ij} = \text{winding number of } \gamma_i \text{ around } \gamma_j$

Claim: the 2-form below is identically 0 at every point:

$P_{ijk} = \left( \frac{dz_i - dz_j}{z_i - z_j} \right) \wedge \left( \frac{dz_j - dz_k}{z_j - z_k} \right) + \left( \frac{dz_j - dz_k}{z_j - z_k} \right) \wedge \left( \frac{dz_k - dz_i}{z_k - z_i} \right) + \left( \frac{dz_k - dz_i}{z_k - z_i} \right) \wedge \left( \frac{dz_i - dz_j}{z_i - z_j} \right)$

Proof of claim: would suffice to expand and collect terms.

We can set $z_i - z_j = a$, etc., so this is $\frac{A \wedge B}{ab} + \frac{B \wedge C}{bc} + \frac{C \wedge A}{ca}$.

Clear denominators and note $A + B + C = 0$, $ab + bc + ca = 0$.

$\text{abc}P_{ijk} = cA \wedge B + aB \wedge C + bC \wedge A$

$= (a + b)A \wedge B + aB \wedge (A + B) + b(-A - B) \wedge A$

$= ((a + b) - b)A \wedge B$

$= 0$.

Thus $R_{ijk} = 0$ in $H^2(\mathbb{P}^n, \mathbb{C})$. Since $H^*(\mathbb{P}^n)$ is torsion-free, $R_{ijk} = 0$ in $H^2(\mathbb{P}^n)$.

Corollary to claim: $W_{ij} \mapsto \omega_{ij}$ defines an injection $H^*(\mathbb{P}^n) \hookrightarrow \Omega^*(\mathbb{P}^n, \mathbb{C})$ into the algebra of closed differential forms.

Corollary: $\mathbb{P}^n$ is formal (in the sense of Sullivan theory).