Bott periodicity

Goal: understand topology of $GL_n \mathbb{C}$.
First reduction: Gram-Schmidt gives a deformation retraction of $GL_n \mathbb{C}$ onto $U(n)$, the compact group of $n \times n$ unitary matrices (preserving the inner product on $\mathbb{C}^n$).

Since $U(n+1)$ preserves length, it acts on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$; the stabilizer of a vector is $U(n)$, giving a fiber bundle $U(n) \to U(n+1) \to S^{2n+1}$

$\Rightarrow \pi_i(U(n)) \approx \pi_i(U(n+1))$ for $i < 2n$.

Examples:

$U(1) = S^1$, so $\pi_1(U(1)) = \mathbb{Z}$

$S^1 \to U(2)$ so $\pi_2(U(2)) = 0$ \[ \Rightarrow \pi_1(U(n)) = \mathbb{Z}, \text{ detected by determinant: } U(n) \to S^1 \]

For any Lie group $S^3 \to U(2)$ so $\pi_3(U(2)) = \mathbb{Z}$ \[ \Rightarrow \pi_2(U(n)) = \mathbb{Z}, \text{ true for any Lie group} \]

we see a pattern emerging...

Bott periodicity: $\pi_i(U(n)) \approx \pi_{i+2}(U(n))$ for $n \gg i$.

Let $U = \lim_{n \to \infty} U(n) = U.U(n)$. Fixing $H = C^\infty$, $U$ is unitary transformations of $H$ which are the identity on a co-finite dimensional subspace of $H$.

Bott periodicity $\iff \pi_i(U) \approx \pi_{i+2}(U)$ for all $i \iff U \approx \Omega^2 U$

Since we know $\Omega BU \approx U$ for any group, it suffices to prove that $BU \approx \Omega^2 U$.

Of course this is false: $BU$ is connected, while $\pi_1(\Omega U) = \pi_1(U) = \mathbb{Z}$. But this is easily fixed.

Bott periodicity, rephrased: $\mathbb{Z} \times BU \approx \Omega^2 U$.

Outline of proof:

Let $K = \Omega BU(n)$. $K$ classifies complex vector bundles, so direct sum of vector bundles corresponds to an operation $K \times K \to K$ which we describe later.

We will show that $BK \approx U$.

Then the group completion theorem gives $H_\ast(K)[\mathbb{Q}] \approx H_\ast(\Omega U)$.

But $\pi_0(K) = \mathbb{N}$, so inverting $\pi_0$, group-completes $\pi_0$ to $\mathbb{Z}$, and stabilizes $H_\ast(BU(n))$ to $H_\ast(BU)$:

$H_\ast(K)[\mathbb{Q}] = H_\ast(\Omega BU(n))[\mathbb{Q}] = H_\ast(\mathbb{Z} \times BU)$,

and both $\mathbb{Z} \times BU$ and $\Omega U$ are simply connected

$(\pi_1(BU) = \pi_0(U) = 0)$ 

$(\pi_1(\Omega U) = \pi_2(U) = 0)$

so by the Hurewicz + Whitehead theorems,

a homology equivalence $H_\ast(\mathbb{Z} \times BU) \approx H_\ast(\Omega U)$

is a homotopy equivalence $\mathbb{Z} \times BU \approx \Omega^2 U$, as desired.
Understanding \( K = \prod_{n=1}^{\infty} BU(n) \)

Let \( F_n = \text{space of orthonormal } n\text{-frames in } H \).

\( U(n) \) acts on \( F_n \) with quotient

\( Gr_n = \text{space of } n\text{-planes in } H \).

Since \( F_n \) is contractible, we have \( Gr_n = F_n / U(n) = BU(n) \)

Thus \( K = \prod_{n=1}^{\infty} BU(n) \) is the space of finite-dimensional subspaces of \( H \).

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What is our operation \( K \times K \to K \)? It suffices to define it on each component

\[ BU(n) \times BU(m) \to \prod_{k=0}^{\infty} BU(k) \]

We take the map

\[ BU(n) \times BU(m) \to BU(n+m) \]

induced by the inclusion

\[ U(n) \times U(m) \to U(n+m) \]

\[
\begin{bmatrix} \ast & 0 \\ 0 & \ast \end{bmatrix}
\]

Note for later:

Let \( Gr_{n,m} = \text{space of pairs } (V_1, V_2) \text{ with } V_1 \text{ on an } n\text{-plane in } H, \text{ with } V_1 \text{ and } V_2 \text{ orthogonal} \) and \( V_2 \text{ on an } m\text{-plane in } H \),

Then \( Gr_{n,m} = F_{n+m} / U(n) \times U(m) \), so \( Gr_{n,m} = B(U(n) \times U(m)) \)

and the map \( Gr_{n,m} \to Gr_{n+m} \) realizes the map \( B(U(n) \times U(m)) \to BU(n+m) \)

\( (V_1, V_2) \mapsto V_1 \oplus V_2 \)

Induced by inclusion \( U(n) \times U(m) \to U(n+m) \)

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The relaxation principle suggests that we take as a model for \( BK \)

the space of all \((V_1, V_2, \ldots, V_n)\) finite subsets of \((0, 1)\) labeled by subspaces \( V_i \) of \( H \)

topologized to vanish at the boundary

each element is determined by \( n \geq 0 \) subspaces \( V_1, \ldots, V_n \in K \)

and by the separations \((t_0, t_1, \ldots, t_n)\) satisfying \( t_i > 0, \ t_0 + \ldots + t_n = 1 \)

so as a set this space is \( BK = \prod_{n \geq 0} K \times \cdots \times K \times \Delta^n \)

\[ \begin{array}{c}
\text{V}_1 \\
\text{V}_2 \\
\vdots \\
\text{V}_n \\
\hline
(t_0, \ldots, t_n)
\end{array} \]

but we need to topologize so that

as \( t_i \to 0 \) and \( V_1 \) and \( V_2 \) come together, we replace them by \( V_1 \vee V_2 \in K \).

Problem: to do this explicitly, we need a concrete realization of the map \( \ast : Gr_n \times Gr_m \to Gr_{n+m} \), which takes some work.

Plus, even given this, it takes a lot of bookkeeping to make it associative (as \( V_i, V_j, \) and \( V_k \) come together simultaneously, what do we replace them with?).
Solution: restrict to work with subspace on which direct sum has a nice geometric model.

Let $Y = \text{space of all } (V_1, V_2, \ldots, V_n) \text{ finite subsets of } (0,1) \text{ labeled by } V_i < H$ s.t. $V_i$ and $V_j$ are orthogonal, topologized so that as $V_i$ and $V_{i+1}$ come together ($t_i \to 0$), we replace them by their span $V_i \oplus V_{i+1}$, and $V_i$ and $V_n$ can disappear at the boundary ($t_n \to 0$).

Claim: $Y \simeq BK$.

The space of $n$-simplices in $Y$ is the space of sequences $V_1, \ldots, V_n$ with $V_i < H$ and $V_i$ orthogonal to $V_j$.

Ex: for a 2-point set we have an $n$-plane $V_i$ and an $m$-plane $V_j$ that are orthogonal.

The space of such pairs is $Gr_n, m = B(U(n) \times U(m))$.

Note that the projections $Gr_n \to Gr_n, m$ induce a homotopy equivalence $Gr_n, m \simeq Gr_n \times Gr_m$.

Thus $Y = \bigcup_{n \geq 0} \bigcup_{k_1, \ldots, k_n} Gr_{k_1, \ldots, k_n} \times \Delta^n / \sim$.

While $BK = \bigcup_{n \geq 0} K \times \cdots \times K \times \Delta^n / \sim$.

Since $Gr_{k_1, \ldots, k_n} \simeq Gr_{k_1} \times \cdots \times Gr_{k_n}$, the spaces of $n$-simplices are homotopy equivalent, and this implies that $Y \simeq BK$.

It remains to show that $Y \simeq U$.

Instead of points in $\bigcup_{n \geq 0} K \times \cdots \times K \times \Delta^n$, think of $Y$ as points in $*$.

So a point in $Y$ is a finite collection of complex numbers $\lambda_i \neq 1$ in the unit circle, and for each $\lambda_i$, a finite-dimensional subspace $V_i < H$ which are orthogonal.

But the spectral theorem states that every unitary matrix is unitarily diagonalizable with eigenvalues on the unit circle and eigenspaces orthogonal.

As two eigenvalues come together, the corresponding eigenspaces are combined; as an eigenvalue goes to 1, we need no longer keep track of its eigenspace.

Conclusion: Our model $Y = BK = B(U \times U(n))$ is not only homotopy equivalent to $U$, it is exactly homeomorphic to $U$. 