Twisted K-theory and groupoids

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Introduction

Theorem (Atiyah, Segal)

[Completion theorem] Let G be a compact Lie group and X a finite G-space. Then:

$$K^*_G(X)^{\wedge}_{I_G} \cong K^*\left(\frac{X \times EG}{G}\right)$$

where $I_G = Ker[R(G) \overset{Aug}{\to} \mathbb{Z}] = Ker[K^*_G(pt) \to K^*(pt)]$ *is the augmentation ideal.*

- Analogue for groupoids?

Introduction

Theorem (C)

Let $\mathcal{G}$ be a finite BC Lie groupoid, X is a finite $\mathcal{G}$-CW-complex and P a $\mathcal{G}$-stable projective bundle on X. Then, we have an isomorphism of $K^*_{\mathcal{G}}(G_0)$-modules:

$$ {}^{P}K^n_{\mathcal{G}}(X)^{\wedge}_{I_{\mathcal{G}}} \longrightarrow {}^{P\times_\pi E\mathcal{G}}K^n_{\mathcal{G}}(X \times_\pi E\mathcal{G}) $$

where $I_{\mathcal{G}} = Ker[K^*_{\mathcal{G}}(G_0) \to K^*(G_0)]$.

- In particular $K^*_{\mathcal{G}}(G_0)^{\wedge}_{I_{\mathcal{G}}} \cong K^*(B\mathcal{G})$

Lie groupoids

Definition

A Lie groupoid is a category $\mathcal{G} = (G_0, G_1)$ in which every arrow is invertible and:

- The space of objects G_0 and the space of arrows G_1 are smooth manifolds.
- The structure maps are smooth. Moreover, s and t are submersions.

$$G_1 \xrightarrow{s,t} G_0 \xrightarrow{u} G_1 \xrightarrow{i} G_1$$

$$G_1 \times_{s,t} G_1 \xrightarrow{m} G_1$$

Groupoid actions

Definition

A $\mathcal{G}$-action on X is given by two maps, the anchor map $\pi : X \to G_0$ and $\mu : X \times_{G_0} G_1 \to X$. The latter map is defined on pairs (x, g) with $\pi(x) = t(g)$ and written $\mu(x, g) = x \cdot g$. They must satisfy the conditions:

- $\pi(x \cdot g) = s(g)$
- $x \cdot u(\pi(x)) = x$
- $(x \cdot g) \cdot h = x \cdot (gh)$ when the operations are defined.

Action groupoids

Example

Let M be a smooth manifold with a smooth action of a Lie group G.
$\mathcal{G} = G \rtimes M$ action groupoid
$G_0 = M$
$G_1 = M \times G$
An action of $\mathcal{G}$ on X is given by an action of G on X and a G-equivariant map $X \to M$

Vector bundles

Definition

Let $\mathcal{G}$ be a groupoid. A $\mathcal{G}$-vector bundle on a $\mathcal{G}$-space X is a vector bundle $p : V \longrightarrow X$ such that V is a $\mathcal{G}$-space with fibrewise linear action and p is a $\mathcal{G}$-equivariant map.

Definition

We say V is extendable if there is a $\mathcal{G}$-vector bundle $W \longrightarrow G_0$ such that V is a direct summand of $\pi^* W$.

Bredon-compatibility

Definition

An n-dimensional $\mathcal{G}$-cell is a space of the form $D^n \times U$ where U is a $\mathcal{G}$-space such that $\mathcal{G} \rtimes U$ is weakly equivalent to an action groupoid corresponding to a proper action of a compact Lie group G on a finite G-CW-complex.

Definition

A groupoid $\mathcal{G}$ is Bredon-compatible if given any $\mathcal{G}$-cell U, all $\mathcal{G}$-vector bundles on U are extendable.

Representability

Theorem

If $\mathcal{G}$ is a Bredon-compatible Lie groupoid, the groups $K^n_{\mathcal{G}}(X, A)$ defined using extendable vector bundles form a $\mathbb{Z}/2$-graded multiplicative cohomology theory on the category of finite $\mathcal{G}$-CW-pairs.

Theorem

Let $\mathcal{G}$ be a Bredon-compatible finite Lie groupoid, X a finite $\mathcal{G}$-CW-complex and H a stable representation of $\mathcal{G}$, then:

$$K^n_{\mathcal{G}}(X) = \begin{cases} [X, \mathit{Fred}'(H)]^{ext}_{\mathcal{G}} & \text{if } n \text{ is even} \\ [X, \Omega \mathit{Fred}'(H)]^{ext}_{\mathcal{G}} & \text{if } n \text{ is odd} \end{cases}$$

Twisted K-theory

Definition

Let $P \to X$ be a $\mathcal{G}$-stable projective bundle. The $\mathcal{G}$-equivariant twisted K-theory of X with twisting P is the group of extendable homotopy classes of extendable $\mathcal{G}$-equivariant sections of $\mathit{Fred}'(P)$ and we denote it by ${}^{P}K_{\mathcal{G}}(X)$.

Theorem

The groups ${}^{P}K_{\mathcal{G}}^{n}(X)$ define a $\mathbb{Z}/2$-graded cohomology theory on the category of finite $\mathcal{G}$-CW-complexes with $\mathcal{G}$-stable projective bundles.

Universal spaces

- $E^n\mathcal{G} = G_1 *_s \overset{n}{\ldots} *_s G_1$
- $\pi(\Sigma_{i=1}^n \lambda_i g_i) = s(g_1)$
- $(\Sigma_{i=1}^n \lambda_i g_i) \cdot g = \Sigma_{i=1}^n \lambda_i g_i g$
- Subspace topology from $G_1 * \overset{n}{\ldots} * G_1$.

- $E\mathcal{G} = \lim E^n\mathcal{G}$ (Universal $\mathcal{G}$-space)
- $\mathcal{G}$ acts freely on $E\mathcal{G}$
- $E\mathcal{G}/\mathcal{G} = B\mathcal{G}$, the classifying space of $\mathcal{G}$.

Example

If $\mathcal{G} = G \rtimes M$, then $E\mathcal{G} = M \times EG$ and $B\mathcal{G} = (M \times EG)/G$

The completion map

Augmentation map:

$$K^*_{\mathcal{G}}(G_0) \to K^*_{\mathcal{G}}(E^n\mathcal{G}) \cong K^*(B^n\mathcal{G}) \to K^*(G_0)$$

We call its kernel the augmentation ideal and denote it by $I_{\mathcal{G}}$

The first map factors through $I^n_{\mathcal{G}}$, that is, we have a map of pro-rings:

$$\{K^*_{\mathcal{G}}(G_0)/I^n_{\mathcal{G}}\} \xrightarrow{F_{\mathcal{G}}} \{K^*(B^n\mathcal{G})\}$$

By naturality, for any $\mathcal{G}$-space X we obtain a map of pro-algebras:

$$\{K^*_{\mathcal{G}}(X)/I^n_{\mathcal{G}}K^*_{\mathcal{G}}(X)\} \to \{K^*(X \times_{\pi} E^n\mathcal{G}/\mathcal{G})\}$$

The completion theorem

Theorem

If $\mathcal{G}$ and $\mathcal{H}$ are weakly equivalent, then $F_{\mathcal{G}}$ is an isomorphism if and only if $F_{\mathcal{H}}$ is.

- If $\mathcal{G} = G \rtimes M$ where G is a compact Lie group and M is a finite G-CW-complex, $F_{\mathcal{G}}$ is an isomorphism.
- If U is a $\mathcal{G}$-cell, then $F_{\mathcal{G} \rtimes U}$ is an isomorphism.

Theorem

Let $\mathcal{G}$ be a finite BC Lie groupoid, X be a finite $\mathcal{G}$-CW-complex and P a $\mathcal{G}$-stable projective bundle on X. Then we have an isomorphism of $K^*_{\mathcal{G}}(G_0)$-modules:

$$^{P}K^n_{\mathcal{G}}(X)^{\wedge}_{I_{\mathcal{G}}} \longrightarrow {}^{P\times_\pi E\mathcal{G}}K^n_{\mathcal{G}}(X \times_\pi E\mathcal{G})$$

Proper actions

Consider $\mathcal{G} = S \rtimes \underline{E}S$, where $\underline{E}S$ is the classifying space for proper actions, that is, a S-CW-complex with compact isotropy and such that $(\underline{E}S)^H$ is contractible when H is compact.

When is $\mathcal{G}$ Bredon-compatible?

- Finite groups. [Atiyah,Segal]
- Compact Lie groups. [Atiyah,Segal]
- Discrete groups. [Luck]
- Almost compact groups. [Phillips]
- Closed subgroups of $GL_n(\mathbb{R})$. [Phillips]
- Pro-discrete groups, for example, $SL_2(\mathbb{Z}_p^\wedge)$ [Sauer]

Kac-Moody groups

- A case when $\mathcal{G}$ is not Bredon-compatible in general is for Kac-Moody groups. Essentially all their finite-dimensional representations are trivial.
- There are some interesting infinite-dimensional representations called dominant representations, which allow us to define dominant K-theory [N. Kitchloo]. There seems to be a completion theorem as well.

Open questions:

- Thom isomorphism?
- Atiyah-Hirzebruch spectral sequence?
- Other models for $\mathcal{G}$-CW-complexes?
- Other groupoids that are Bredon-compatible?
- Borcherds groups?

In arxiv.org:

- Equivariant K-theory, groupoids and proper actions, Jose Cantarero.
- Twisted equivariant K-theory, groupoids and proper actions, Jose Cantarero.