Maxwell's Equations

by Daniel Bump for Math 52

The purpose of this handout is to give some examples showing the application of the two main theorems in Vector Calculus as we've presented it—Gauss' Divergence Theorem and Stoke's Theorem. We will give a self-contained exposition of the basic principles of electricity and magnetism.

The fact that electricity and magnetism are related phenomena was first demonstrated by Oersted, who showed in 1820 that a *changing* electric current can deflect a magnetized needle. He placed the needle near a wire, through which a current was run from a battery. When the battery was turned on or off, the needle jumped. This discovery was immediately the cause of a great deal of experimentation, by Ampère and others, but above all by Faraday.

Michael Faraday (1791-1867) was the unusual combination of a first-rate experimental physicist and an important theoretician. He thought of the electric and magnetic fields in terms of "lines of force," which are (in the language we've been following) the flow lines of the electric and magnetic fields. Faraday did many important experiments elucidating the nature of electricity and magnetism.

The laws of electricity and magnetism may be formulated in either an "integral" form or a "differential" form. The integral form is closer to intuitive understanding, but the differential form is more powerful from the theoretic point of view. Faraday arrived at a complete theory of electricity and magnetism in integral form. He demonstrated the phenomenon of electromagnetic induction in 1831, and this was essentially the final piece to the puzzle.

James Clerk Maxwell (1831-1879) recast the laws of electricity and magnetism in differential form. Passing between the integral form and the differential form involves notions of vector calculus, particularly Gauss' Divergence Theorem and Stokes' Theorem. The differential form of the theory is encompassed in *Maxwell's equations.* These are partial differential equations which have wave-like solutions. Thus Maxwell showed that electromagnetic waves should exist, and he explained light as a form of electromagnetic radiation.

Maxwell's equations also led directly to the theory of special relativity in the work of Lorentz and Einstein.

Current, Charges, and Fields

We will chose our units of time and distance so that the velocity of light is equal to 1. We will denote the electric field by $\mathbf{E}$ and the magnetic field by $\mathbf{B}$. These may be characterized as follows. If a particle with charge q is located at some point P, and is moving with velocity $\mathbf{v}$, then the electromagnetic force on the particle is

$$q\big(\mathbf{E}(P) + \mathbf{v} \times \mathbf{B}(P)\big).$$

A consistent theory can be given in which magnetic particles, known as *monopoles*, exist. A magnetic monopole with charge q and velocity $\mathbf{v}$ would experience a force of $q\big(\mathbf{B}(P) - \mathbf{v} \times \mathbf{E}(P)\big)$. Thus the roles of the electric and magnetic fields would be reversed. Monopoles have never been found in nature. If the existence of monopoles is assumed, Maxwell's equations become slightly more complex, but also more symmetrical.

We will denote by ρ the density of electric charge distributed through space. Electric charge behaves like a fluid: it is conserved (cannot be created or destroyed) and it can flow. We will denote by $\mathbf{J}$ the vector field describing the electric current—thus $\mathbf{J}(P)$ is the vector indicating the direction and rate of flow of electric charge at the point P.

Suppose that D is a bounded domain in $\mathbb{R}^3$, with boundary ∂D. Then $\int_{\partial D} \mathbf{J} \cdot \mathbf{n}\, \mathbf{dS}$ is the current flux across the boundary, which is evidently the rate at which charge is flowing out of D. Since the total charge inside D is $\int_D \rho(\mathbf{x})\, dV_{\mathbf{x}}$, we have

$$\int_{\partial D} \mathbf{J} \cdot \mathbf{n}\, \mathbf{dS} = -\frac{\partial}{\partial t} \int_D \rho(\mathbf{x})\, dV_{\mathbf{x}} = \int_D -\frac{\partial \rho}{\partial t}(\mathbf{x})\, dV_{\mathbf{x}}.$$

On the other hand, by Gauss' Divergence Theorem,

$$\int_{\partial D} \mathbf{J} \cdot \mathbf{n}\, \mathbf{dS} = \int_D \operatorname{div}(\mathbf{J})\, dV_{\mathbf{x}}.$$

Thus

$$\int_D \operatorname{div}(\mathbf{J})\, dV_{\mathbf{x}} = \int_D -\frac{\partial \rho}{\partial t}\, dV_{\mathbf{x}}.$$

Since this is true for every D, we have

$$\operatorname{div}(\mathbf{J}) = -\frac{\partial \rho}{\partial t}. \tag{1}$$

Therefore the divergence of the electric current is the rate of change of electric charge.

We have already explained how in the theory of gravitation the divergence of the gravitational field is proportional to the mass density function:

$$\operatorname{div}(\mathbf{G}) = -4\pi\mu,$$

where $\mathbf{G}$ is the gravitational field and μ is the mass density. The same is true of the electric field:

$$\operatorname{div}(\mathbf{E}) = 4\pi\rho. \tag{2}$$

The proof, based on Columb's Law, is the same as in the gravitational force, but since the electric field arises from a repulsive force (in the case of like charges) and the gravitational force from an attractive one, the sign is positive in (2).

Similarly the divergence of the magnetic field should equal the density of magnetic charge. However since as far as we know magnetic monopoles do not exist, this equation simply states

$$\mathrm{div}(\mathbf{B}) = 0. \tag{3}$$

Changing Magnetic Flux

It is an empirical fact that if the magnetic field is zero or unchanging, then the curl of the electric field is zero. We may illustrate this with a loop γ of wire. At each point in the wire, the electrons will experience a force proportional to the electric field. Since the electrons are constrained to move in the direction of the wire, the net force along the wire is proportional to the circulation of the electric field around the loop, given by the line integral:

$$\int_\gamma \mathbf{E} \cdot d\mathbf{x}.$$

(The protons in the wire will experience a similar but opposite force. The electrons can move, the protons cannot, because they are confined to their atomic nuclei which are relatively immobile.) It is observed that unless there is a changing magnetic field in the picture, a voltmeter spliced into the loop, which measures this net force, is not deflected. Therefore

$$\int_\gamma \mathbf{E} \cdot d\mathbf{x} = 0.$$

We may use Stokes' Theorem to reinterpret this result in the form

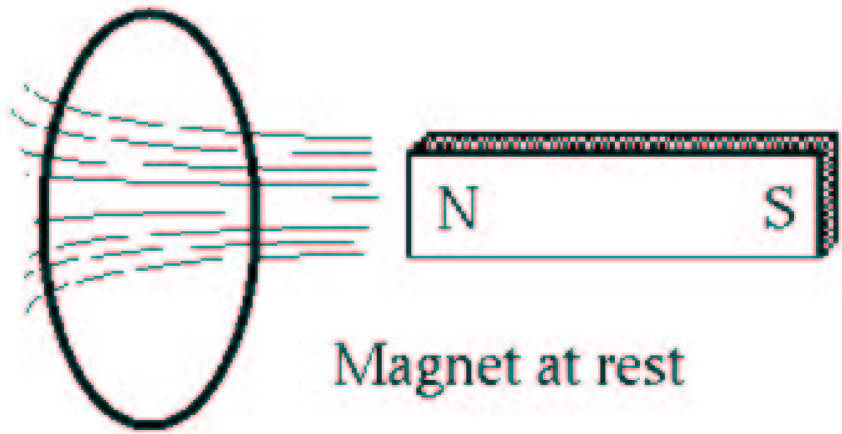
$$\int_S \mathbf{curl}(\mathbf{E}) \cdot \mathbf{n}\, \mathbf{dS} = 0,$$

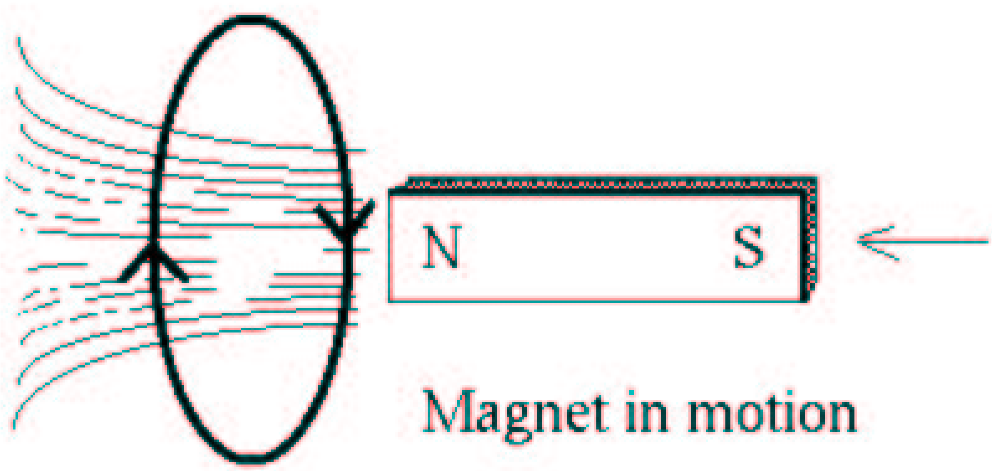
where the integral is over a surface spanned by the loop. Since this is true regardless of the location of the loop, we infer that

$$\mathbf{curl}(\mathbf{E}) = 0.$$

Thus at least in this experiment the electric field is conservative.

Now let us take a magnet and move it into the proximity of the loop.





It is observed that while the magnet is moving, a voltage appears, and a current flows in the loop. When the magnet stops, the voltage and the current disappear. We have duplicated Faraday's construction of the first electric generator. We note that although the agent which caused the current was a magnet, the current in the loop must be attributed to an electric field, not a magnetic one. Indeed, each moving electron may experience a force due to the magnetic field. But the direction of such a force (given by $q\mathbf{v} \times \mathbf{B}$) will be perpendicular to the direction of motion of the electron—since the electron is constrained to move in the wire, perpendicular to the wire. Since the electron cannot move in the direction of the force, the magnetic field does not directly cause the current which is observed in the loop. We are seeing the effect of an electric field, caused by the motion of the magnet.

As the magnet is moved towards the loop, the flux of the magnetic field $\mathbf{B}$ through the loop increases. By experiments such as this, Faraday discovered that the voltage on the wire is proportional to the rate of change in the magnetic flux through the surface S. Thus

$$\int_\gamma \mathbf{E} \cdot d\mathbf{x} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{n}\, \mathbf{dS}.$$

Note that we can use any surface spanned by the loop to measure the magnetic flux. This is because of Gauss' divergence theorem, and (3). If we consider the flux over two different surfaces spanned by the same curve γ, the difference between these two fluxes will be the integral of the divergence of $\mathbf{B}$ over the domain between the two surfaces, and this is zero.

By Stokes' Theorem, we have

$$\int_S \mathbf{curl}(\mathbf{E}) \cdot \mathbf{n}\, \mathbf{dS} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{n}\, \mathbf{dS}.$$

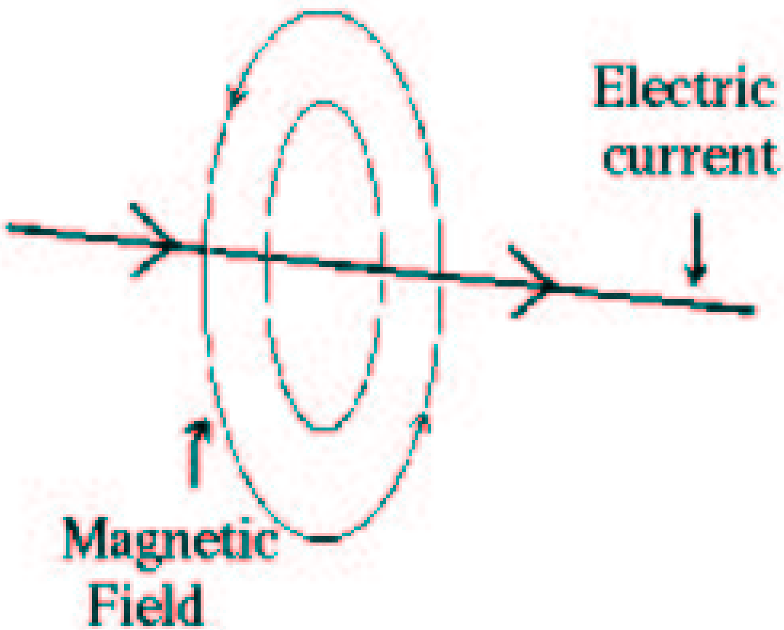
Because this is true for any surface S, we have

$$\mathbf{curl}(\mathbf{E}) = -\frac{\partial \mathbf{B}}{\partial t}. \tag{4}$$

Current Flux

It is an empirical fact that an electric current gives rise to a magnetic field. In order to study this phenomenon we will initially consider the case where the divergence of the electric current is zero. By (1), this means that $\partial\rho/\partial t = 0$. Thus electric charges may be moving ($\mathbf{J} \neq 0$) but the amount of charge moving into any region equals the amount moving out, so that the net amount of charge is zero. This is exactly the meaning of our assumption that $\text{div}(\mathbf{J}) = 0$.

Consider an steady electric current through a wire.



The current is observed to give rise to a magnetic field which circulates around the wire. The circulation of the field around a path surrounding the wire is clearly nonzero—it is found empirically that the circulation is proportional to the current flux through the loop. Let S be a surface spanned by the loop. Then it is found that

$$\int_\gamma \mathbf{B}(\mathbf{x}) \cdot d\mathbf{x} = 4\pi \int_S \mathbf{J} \cdot \mathbf{n}\, \mathbf{dS}.$$

Note that the right side of this equation is independent of the choice of S—this is a consequence of our assumption that $\mathrm{div}(\mathbf{J}) = 0$, since if S_1 and S_2 are two surfaces spanned by the same loop γ, the difference between $\int_{S_1} \mathbf{J} \cdot \mathbf{n}\, \mathbf{dS}$ and $\int_{S_2} \mathbf{J} \cdot \mathbf{n}\, \mathbf{dS}$ equals the integral of the divergence of $\mathbf{J}$ over the region between S_1 and S_2, and this is zero.

By Stokes' Theorem,

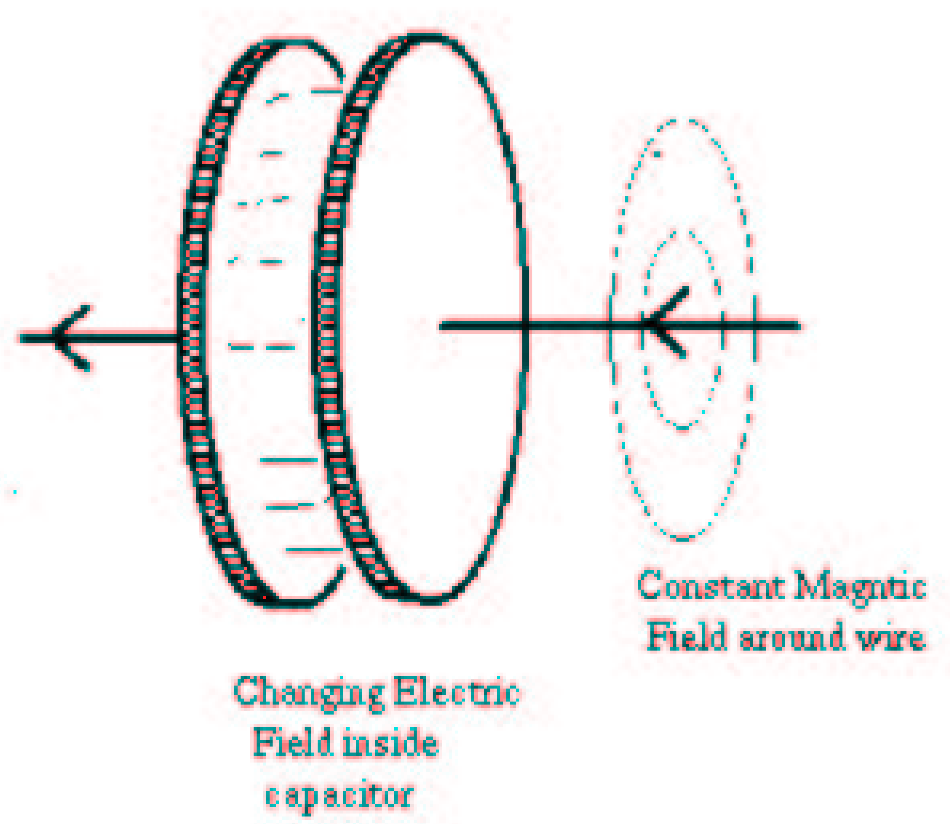
$$\int_\gamma \mathbf{B}(\mathbf{x}) \cdot d\mathbf{x} = \int_S \mathbf{curl}(\mathbf{B}) \cdot \mathbf{n}\, \mathbf{dS}.$$

Therefore $\int_S \big(4\pi\mathbf{J} - \mathbf{curl}(\mathbf{B})\big) \cdot \mathbf{n}\, \mathbf{dS} = 0$, and from this we infer that

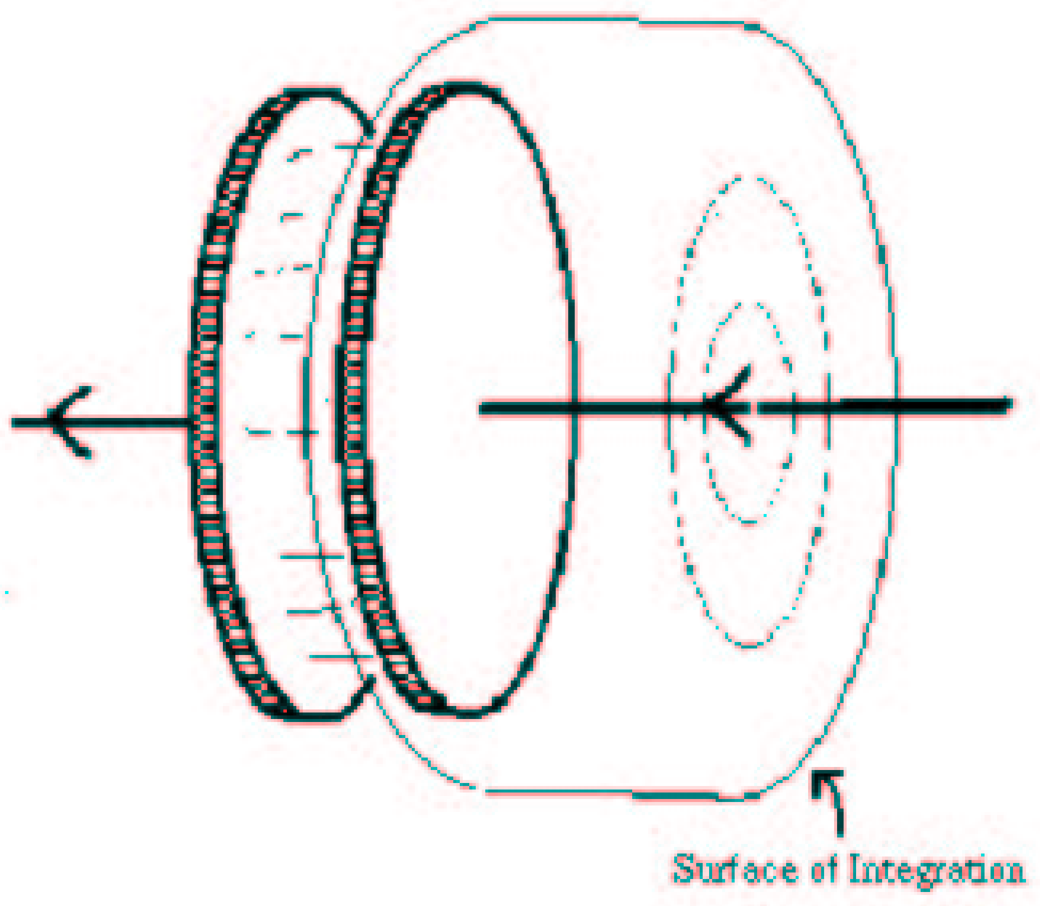
$$\mathbf{curl}(\mathbf{B}) = 4\pi\mathbf{J}. \tag{5}$$

Displacement Current

In the previous section, we assumed that the divergence of **J** is zero, or equivalently, that $\partial\rho/\partial t = 0$. If this is not true, then (5) requires modification. An experiment which illustrates this situation involves a capacitor. This is a device consisting of two thin plates separated by a gap, which can be used to store a voltage. A wire is attached to each plate, and a voltage is placed on the wire. Electric charge flows through the wire, and a positive charge builds up on one plate, a negative charge on the other. The important new feature of this situation is that the electric field between the two plates is steadily increasing as the charges build up.



Now let us integrate **curl**(B) over a surface S which passes between the plates, and also intersects one of the wire loops.



Since the divergence of any curl is zero, Gauss' divergence theorem implies that

$$\int_S \mathbf{curl}(\mathbf{B}) \cdot dS = 0.$$

Near where the surface S meets the wire, we have already seen that there will be a nonzero curl to the magnetic field. There must be some compensating effect inside the capacitor. This suggests that the curl of $\mathbf{B}$ will be nonzero where the electric field is increasing, even if there is no current. This thought is strengthened by the observation that

$$\frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{J}$$

is divergence-free, a property that the curl of any vector field must have. Indeed,

$$\operatorname{div}\left(\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}\right) = \frac{\partial}{\partial t}\operatorname{div}(\mathbf{E}) + 4\pi \operatorname{div}(\mathbf{J}) = 4\pi \frac{\partial \rho}{\partial t} - 4\pi \frac{\partial \rho}{\partial t} = 0.$$

Thus we expect the modification of (5) to have the form

$$\mathbf{curl}(\mathbf{B}) = \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{J} \qquad (6).$$

Maxwell called the first term on the right side of (6) the *displacement current.*

Maxwell's Equations

Equations (2), (3), (4) and (6) are *Maxwell's Equations.* To recapitulate, they state:

$$\mathrm{div}(\mathbf{E}) = 4\pi\rho, \tag{2}$$

$$\mathrm{div}(\mathbf{B}) = 0, \tag{3}$$

$$\mathbf{curl}(\mathbf{E}) = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4}$$

and

$$\mathbf{curl}(\mathbf{B}) = \frac{\partial \mathbf{E}}{\partial t} + 4\pi\mathbf{J} \qquad (6).$$

If magnetic monopoles exist, equations (3) and (4) must be modified—the resulting equations are symmetrical in $\mathbf{E}$ and $\mathbf{B}$.

Maxwell's equations also become symmetrical when we restrict ourselves to the case of the electromagnetic field in empty space. Thus if we assume that ρ and $\mathbf{J}$ are zero, we have the equations:

$$\mathrm{div}(\mathbf{E}) = \mathrm{div}(\mathbf{B}) = 0, \tag{7}$$

$$\mathbf{curl}(\mathbf{E}) = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \mathbf{curl}(\mathbf{B}) = \frac{\partial \mathbf{E}}{\partial t}. \tag{8}$$

Maxwell's equations are a bit more removed from the experimental data which led to them than the integral forms, which are close to Faraday's understanding. However they have a great advantage of allowing us to understand light as *electromagnetic radiation*, a great achievement of Maxwell. To see this, let us decompose the electric field into components: $\mathbf{E} = (E_1, E_2, E_3)$. We wish to show that each component satisfies a certain partial differential equation, the *wave equation*, and solutions to which may be expected to behave like waves in a fluid. Then (8) implies that

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\partial}{\partial t}\mathbf{curl}(\mathbf{B}) = \mathbf{curl}\left(\frac{\partial \mathbf{B}}{\partial t}\right) = -\mathbf{curl}\big(\mathbf{curl}(\mathbf{E})\big).$$

Now let us consider the curl of the curl of $\mathbf{E}$. This equals:

$$\left(\frac{\partial}{\partial y}\left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x}\right), \ldots, \ldots\right),$$

where I've omitted the second two components. Making use of (7), the divergence of $\mathbf{E}$ is zero, so

$$\frac{\partial^2 E_2}{\partial y\,\partial x} + \frac{\partial^2 E_3}{\partial z\,\partial x} = -\frac{\partial^2 E_1}{\partial x^2}.$$

Since $\partial^2 E_1/\partial t^2$ is negative the first component of $\mathbf{curl}\big(\mathbf{curl}(\mathbf{E})\big)$, we see that E_1 is a solution to the partial differential equation

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}. \tag{7}$$

Equation (7) is known as the *wave equation*. It models wave motion in a 3-dimensional medium—for example, this is the differential equation satisfied by air pressure; waves in air pressure account for sound.

To get some feeling for the nature of solutions to (7), let f be some real-valued function. Then

$$\phi(x, y, z, t) = f(x - t)$$

is a solution to (7), because

$$\frac{\partial^2 \phi}{\partial t^2} = f''(x - t) = \frac{\partial^2 \phi}{\partial x^2},$$

while the other partial derivatives vanish. This solution is a wave moving from left to right. Similarly,

$$\phi(x, y, z, t) = f(x + t)$$

is a wave moving from right to left. Wave solutions to (7) can move in any direction, always with velocity 1 given our normalizations. Maxwell realized that the electromagnetic field had wave-like solutions. On calculating the velocity of these solutions, and finding it numerically consistent with the velocity of light, he came to realize that light is a form of electromagnetic radiation. After the death of Maxwell, Hertz succeeded in producing radio waves, which are electromagnetic waves of lower frequency than light, thus firmly establishing Maxwell's theories.

Lorentz and Einstein, reflecting on the fact that Maxwell's equations do not seem to be invariant under changes of coordinates, were led to the theory of special relativity.

Further Reading

I highly recommend Purcell's *Electricity and Magnetism* from the Berkeley Physics Series. Also good are the Feynmann *Lectures on Physics*. For historical facts, I consulted the *Encyclopedia Brittanica*.