

## Selected Homework Solutions

Section 15.5 of Edwards and Penny:

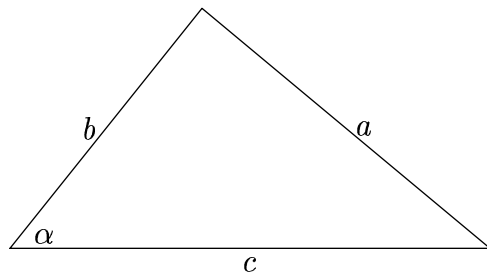
**41.** Consider a homogeneous thin spherical shell  $S$  of radius  $a$  centered at the origin, with density  $\delta$  and total mass  $M = 4\pi a^2\delta$ . A particle of mass  $m$  is located at the point  $(0, 0, c)$  with  $c > a$ . Use the method of problem 41 of Sections 14.7 to show that the gravitational force of attraction between the particle and the spherical shell is

$$F = \int \int_S \frac{Gm\delta}{w^2} dS = \frac{GMm}{c^2}.$$

If you have the most recent version of the book, the relevant problem is 47 in 14.7. The problems in 14.7 were renumbered and this reference was not updated.

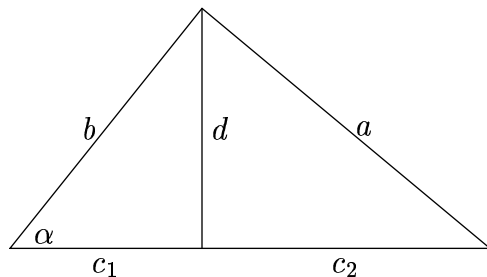
In other words, the force is the same as if the mass were entirely concentrated in the origin. This is easy to prove if we use the divergence theorem and a symmetry argument. But we have to do this the hard way.

First, we need to recall the law of cosines. Consider the following triangle:



**Figure 1.** Law of cosines:  $a^2 = b^2 + c^2 - 2bc \cos(\alpha)$ .

The law of cosines states that  $a^2 = b^2 + c^2 - 2bc \cos(\alpha)$ . To prove this, we can drop a perpendicular from the top vertex to the bottom, splitting  $c$  into two parts:



**Figure 2.** Proving the Law of cosines.

By the Pythagorean theorem,  $d^2 + c_1^2 = b^2$  and  $d^2 + c_2^2 = a^2$ . Subtracting these two equations,

$$(1) \quad a^2 - c_2^2 = b^2 - c_1^2.$$

On the other hand, squaring the equation  $c = c_1 + c_2$  and rearranging,

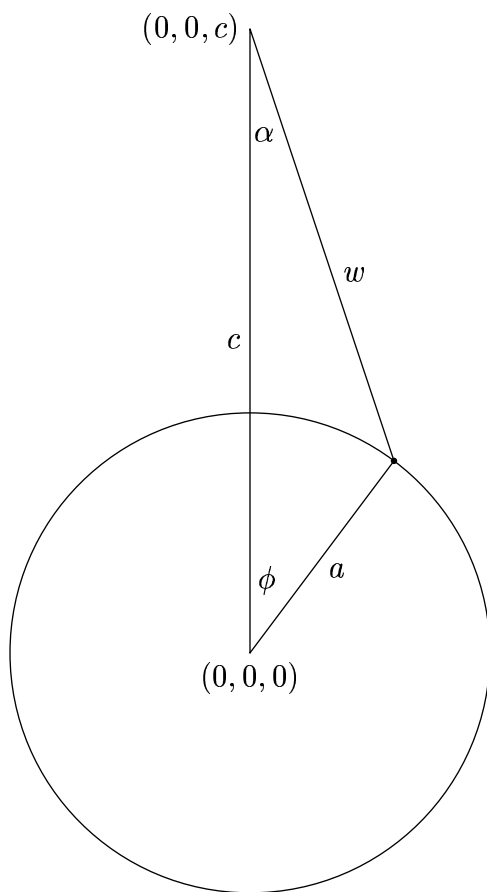
$$(2) \quad c_2^2 = c^2 - c_1^2 - 2c_1c_2.$$

Adding (1) and (2),

$$a^2 = b^2 + c^2 - 2c_1^2 - 2c_1c_2 = b^2 + c^2 - 2c_1(c_1 + c_2).$$

Now  $c_1 = b \cos(\alpha)$  and  $c_1 + c_2 = c$ , so we are done.

Now following the suggestion in the book, we use spherical coordinates, so  $\rho = a$  on the surface of the sphere.



**Figure 3.** Cross section of the sphere.

With a unit point mass at the marked point on the sphere in Figure 3, the magnitude of the force at  $(0, 0, c)$  is  $F = Gm/w^2$ . We are really only interested in the  $z$  component  $F_z$  of the

force since by symmetry, the *net* force contributed by all points of the sphere is vertically downward in the  $z$  direction. If we integrate the force vector over the entire sphere we get the force, and since only its  $z$ -component will be nonzero, we can just integrate  $F_z$ . Clearly  $F_z = F \cos(\alpha)$  and it is this that we must integrate. We must integrate

$$\int \int_S F_z dS = \int \int_S \delta F_z dS = \int_0^{2\pi} \int_0^\pi \cos(\alpha) \frac{Gm\delta}{w^2} a^2 \sin(\phi) d\phi d\theta.$$

This takes into account the fact that the mass density of the sphere is  $\delta$ , and that the element of surface area in spherical coordinates is  $\rho^2 \sin(\phi) d\phi d\theta$ , and  $\rho = a$ .

Now, following the suggestion in the book, we replace the coordinate  $\phi$  by  $w$  as in Figure 3. By the law of cosines  $w^2 = a^2 + c^2 - 2ac \cos(\phi)$ . Since  $a$  and  $c$  are constant in this problem we may differentiate to get

$$2w dw = 2ac \sin(\phi) d\phi.$$

Moreover by the law of cosines  $a^2 = w^2 + c^2 - 2wc \cos(\alpha)$ , so

$$\cos(\alpha) = \frac{w^2 + c^2 - a^2}{2wc}.$$

Noting that  $w$  varies from  $c - a$  (north pole) to  $c + a$  (south pole) we get

$$2\pi Gm\delta \int_{c-a}^{c+a} \frac{a^2}{w^2} \frac{w^2 + c^2 - a^2}{2wc} \frac{w}{ac} dw.$$

The integral is now in a form that it can be done easily. We get

$$\frac{\pi Gma\delta}{c^2} \left[ w - \frac{c^2 - a^2}{w^2} \right]_{w=c-a}^{w=c+a}.$$

After a bit of algebra this equals

$$\frac{\pi Gma\delta}{c^2} \times 4a = \frac{4\pi a^2 m\delta}{c^2}.$$

Since the total mass  $M$  is  $4\pi a^2 \delta$  (surface area times mass density) we are done.

Of course this result can be gotten more easily using the divergence theorem.

Section 15.6:

**5.** *Verify the divergence theorem by direct computation of both the surface integral and the triple integral of the divergence theorem, when  $\mathbf{F} = (x + y)\mathbf{i} + (y + z)\mathbf{j} + (z + x)\mathbf{k}$ , and  $S$  is the surface of the tetrahedron bounded by the three coordinate planes and the plane  $x + y + z = 1$ .*

First let us compute the flux through the plane  $x + y + z = 1$  bounded by  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$ . We parametrize

$$x = u, \quad y = v, \quad z = 1 - u - v.$$

Then if  $\mathbf{n}$  is the unit normal and  $dS$  is the element of surface area, we recall that in

$$\begin{aligned} \mathbf{n} dS &= \left[ \frac{\partial(y, z)}{\partial(u, v)} \mathbf{i} + \frac{\partial(z, x)}{\partial(u, v)} \mathbf{j} + \frac{\partial(x, y)}{\partial(u, v)} \mathbf{k} \right] du dv = \\ &= \left[ \left( \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k} \right) \times \left( \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k} \right) \right] du dv. \end{aligned}$$

In the case at hand, this equals

$$[(\mathbf{i} - \mathbf{k}) \times (\mathbf{j} - \mathbf{k})] du dv = (\mathbf{i} + \mathbf{j} + \mathbf{k}) du dv.$$

Dotting this with  $\mathbf{F}$ , the flux through this face is

$$\int_0^1 \int_0^{1-u} ((x + y) + (y + z) + (z + x)) dv du.$$

On this face,  $2x + 2y + 2z = 2$ , so this is twice the area of the triangle with vertices  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$  in the  $(u, v)$  plane, or just 1.

Next we calculate the flux through the bottom face. This is the triangle with vertices  $(0, 0, 0)$ ,  $(1, 0, 0)$  and  $(0, 1, 0)$ . On this face,  $\mathbf{n} = -\mathbf{k}$ , the downward unit vector, so  $\mathbf{F} \cdot \mathbf{n} = -(z + x)$ . Moreover,  $z = 0$  on this face, so we just have to integrate  $-x$  over this triangle. This flux equals

$$\int_0^1 \int_0^{1-x} (-x) dy dx = - \int_0^1 x(1 - x) dx = -\frac{1}{6}.$$

The flux over the remaining two faces is the same so the total flux is

$$1 - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} = \frac{1}{2}.$$

To verify the divergence theorem, the divergence of this vector field is

$$\frac{\partial}{\partial x}(x + y) + \frac{\partial}{\partial y}(y + z) + \frac{\partial}{\partial z}(z + x) = 3.$$

So the integral of the divergence is 3 times the volume of the tetrahedron, or  $3 \times \frac{1}{6} = \frac{1}{2}$ .

**22.** Let  $\mathbf{r} = (x, y, z)$  and let  $\mathbf{r}_0 = (x_0, y_0, z_0)$  be a fixed point, and suppose that  $\mathbf{F}(x, y, z) = \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3}$ . Show that  $\text{div}(\mathbf{F}) = 0$  except at the point  $\mathbf{r}_0$ .

See also the handout, *Coulomb Force and Potential*. It is easiest to introduce new variables  $\xi = x - x_0$ ,  $\eta = y - y_0$  and  $\zeta = z - z_0$ , so  $\mathbf{r} - \mathbf{r}_0 = (\xi, \eta, \zeta)$ . Thus

$$\mathbf{F} = \frac{\xi}{(\xi^2 + \eta^2 + \zeta^2)^{3/2}} \mathbf{i} + \frac{\eta}{(\xi^2 + \eta^2 + \zeta^2)^{3/2}} \mathbf{j} + \frac{\zeta}{(\xi^2 + \eta^2 + \zeta^2)^{3/2}} \mathbf{k}.$$

Since  $x_0$ ,  $y_0$  and  $z_0$  are constants,  $\partial/\partial x = \partial/\partial \xi$  etc. and the divergence is

$$\frac{\partial}{\partial \xi} \frac{\xi}{(\xi^2 + \eta^2 + \zeta^2)^{3/2}} + \frac{\partial}{\partial \eta} \frac{\eta}{(\xi^2 + \eta^2 + \zeta^2)^{3/2}} + \frac{\partial}{\partial \zeta} \frac{\zeta}{(\xi^2 + \eta^2 + \zeta^2)^{3/2}}.$$

Differentiating, this is

$$\frac{-2\xi^2 + \eta^2 + \zeta^2}{(\xi^2 + \eta^2 + \zeta^2)^{5/2}} + \frac{\xi^2 - 2\eta^2 + \zeta^2 + \zeta^2}{(\xi^2 + \eta^2 + \zeta^2)^{5/2}} + \frac{\xi^2 + \eta^2 - 2\zeta^2}{(\xi^2 + \eta^2 + \zeta^2)^{5/2}} = 0.$$

Note that problems 24, 25 and 27 of this section were essentially done in class during the week of November 4.