Selected Homework Solutions

Section 15.3 of Edwards and Penny:

30. *Let* $\mathbf{F}(x,y) = (-y\mathbf{i} + x\mathbf{j})/(x^2+y^2)$ *for* x *and* y *not both zero. Calculate the values of*

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

along the both the top and the lower halves of the circle $x^2+y^2=1$ *from* $(1,0)$ *to* $(-1,0)$. *Is there a function* $f = f(x,y)$ *for* x *and* y *not both zero such that* $\nabla f = \mathbf{F}$*? Why or why not?*

Solution. Parametrize the upper arc as follows:

$$x = \cos(t), \qquad y = \sin(t),$$

so if $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ then

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} = -\sin(t)\,dt\,\mathbf{i} + \cos(t)\,dt\,\mathbf{j}.$$

On the circle $x^2+y^2=1$, so

$$\mathbf{F}(x,y) = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2} = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}$$

and

$$\mathbf{F}\cdot\mathbf{T}\,ds = \mathbf{F}\cdot\,d\mathbf{r} = \big(\sin^2(t)+\cos^2(t)\big)\,dt = dt.$$

Thus

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^\pi dt = \pi.$$

This is the integral over the top arc.

Parametrize the bottom arc by:

$$x = \cos(t), \qquad y = -\sin(t).$$

In this case

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} = -\sin(t)\,dt\,\mathbf{i} - \cos(t)\,dt\,\mathbf{j}.$$

Now

$$\mathbf{F}(x,y) = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2} = \sin(t)\mathbf{i} - \cos(t)\mathbf{j}$$

and we have

$$\mathbf{F}\cdot\mathbf{T}\,ds = \mathbf{F}\cdot\,d\mathbf{r} = \big(-\sin^2(t)-\cos^2(t)\big)\,dt = -dt.$$

This integral is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^{\pi} -dt = -\pi.$$

There cannot be a function f such that $\nabla f = \mathbf{F}$, because if there was, the integral over either arc would equal $f(-1,0) - f(1,0)$ by the fundamental theorem for line integrals. They would *both* equal this which is impossible since the two line integrals are different.

31. *Show that if the force field* $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ *is conservative then* $\partial P/\partial y = \partial Q/\partial x$*. Show that the force field of problem 30 satisfies the condition* $\partial P/\partial y = \partial Q/\partial x$ *but nevertheless is not conservative.*

Solution. You could show $\partial P/\partial y = \partial Q/\partial x$ by quoting Theorem 3, but a better answer would be to show this as follows. Since $\mathbf{F}$ is conservative, $\mathbf{F} = \nabla f$ for some scalar field f, and this means that $P = \partial f/\partial x$ and $Q = \partial f/\partial y$. Now

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y\, \partial x}$$

while

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x\, \partial y}.$$

These are equal by a basic property of partial derivatives, the *equality of mixed partials.*

For the vector field of problem 30, we have

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2-x^2}{x^2+y^2}$$

and $\partial P/\partial y$ is the same. But it is shown in problem 30 that the vector field is not conservative.

32. it Suppose that the force field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is conservative. Show that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Solution. Since $\mathbf{F}$ is conservative, $\mathbf{F} = \nabla f$ for some f, so $P = \partial f/\partial x$, $Q = \partial f/\partial y$ and $R = \partial f/\partial z$. Now we proceed as in problem 31:

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial Q}{\partial x},$$

and similarly for the other two parts.

35. *(a) Let* $f(x, y) = \arctan(y/x)$*, which if* $x > 0$ *equals the polar angle* θ *for the point* (x, y)*. Show that*

$$\mathbf{F} = \nabla f = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}.$$

(b) Suppose that $A(x_1, y_2) = (r_1, \theta_1)$ *and* $B(x_2, y_2) = (r_2, \theta_2)$ *are two points in the right half plane* $x > 0$ *and that* C *is a smooth curve from* A *to* B*. Explain why it follows from the fundamental theorem for line integrals that* $\int_C \mathbf{F} \cdot T\, ds = \theta_2 - \theta_1$*.*

(c) Suppose that C_1 *is the upper half of the unit circle from* $(1, 0)$ *to* $(-1, 0)$ *and that* C_2 *is the lower half, oriented also from* $(1, 0)$ *to* $(-1, 0)$*. Show that*

$$\int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds = \pi, \qquad \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds = -\pi.$$

Why does this not contradict the fundamental theorem?

Solution. (a) Recalling that $d \arctan(t)/dt = 1/(1 + t^2)$, it follows from the chain rule, taking $t = y/x$ that

$$\frac{\partial}{\partial x} \arctan(y/x) = \frac{dt}{dx}\frac{d}{dt} \arctan t = \frac{-y}{x^2} \cdot \frac{1}{1 + (y/x)^2} = \frac{-y}{x^2 + y^2},$$

and similarly

$$\frac{\partial}{\partial y} \arctan(y/x) = \frac{1}{x} \cdot \frac{1}{1 + (y/x)^2} = \frac{x}{x^2 + y^2}.$$

Hence the gradient ∇f, where $f = \arctan(y/x)$ is

$$\frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}.$$

(b) Evidently it is intended that the curve C stays in the right half plane $x > 0$, although the problem as worded doesn't explicitly say this. In that case, we can apply the fundamental theorem with $f = \arctan(y/x)$, and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds = f(B) - F(A) = \theta_2 - \theta_1.$$

(c) The computation of the two line integrals was done in problem 30. The fundamental theorem is not contradicted because the vector field is not conservative—there is no function f defined in the whole plane such that $\nabla f = \mathbf{F}$. The function $f = \arctan(y/x)$ from (a) was only defined if $x > 0$, and if we try to extend it to the whole plane, we are not able to do so in a consistent manner.