Selected Homework Solutions

Section 14.3 of Edwards and Penny:

27. *Find the volume of the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $3x+2y+z = 6$.*

First Solution: This is the tetrahedron with vertices at $(0,0,0)$, $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$. Its projection onto the (x,y) plane is the triangle with vertices $(0,0)$, $(2,0)$ and $(0,3)$. Call this region R. The volume in question is the volume above R below the plane $z = 6 - 3x - 2y$, that is

$$\int\int_R (6 - 3x - 2y)\, dA.$$

To write this as an iterated integral, we parametrize R as follows. Clearly x can go from 0 to 2, and with x fixed, y can go from 0 to $3 - 3x/2$, because $y = 3 - 3x/2$ is the line in the (x,y)-plane through $(2,0)$ and $(0,3)$. The volume equals

$$\int_0^2 \int_0^{3-3x/2} (6 - 3x - 2y)\, dy\, dx = \int_0^2 [6y - 3xy - y^2]_{y=0}^{y=3-3x/2}\, dx =$$
$$\int_0^2 [6(3 - 3x/2) - 3x(3 - 3x/2) - (3 - 3x/2)^2]\, dx = \int_0^2 (9 - 9x + 9x^2/4)\, dx =$$
$$[9x - 9x^2/2 + 3x^3/4]_{x=0}^{x=2} = 6.$$

Second Solution: Same result with less computation.

Consider the z-cross section for fixed z. This is a triangle whose area is 0 when $z = 6$, and 3 when $z = 0$. (When $z = 0$, it is the triangle with vertices (0,0), (2,0) and (0,3).) The dimensions of the triangular cross section increase linearly as z decreases from 6 to 0, and since the area of a triangle is proportional to the square of its linear dimensions, the area $A(z)$ of the z-cross section equals $c(6 - z)^2$ for some constant c. Putting $z = 0$ gives the equation $36c = 3$, so $c = \frac{1}{12}$. Thus we get

$$\int_0^6 A(z)\, dz = \int_0^6 \frac{1}{12}(6 - z)^2\, dz.$$

Either expand $(6 - z)^2$ and integrate, or make the substitution $u = 6 - z$, $du = -dz$, to obtain

$$\int_6^0 \frac{1}{12}u^2\, (-du) = \frac{1}{12}\int_0^6 u^2\, du = 6.$$

Problem 35 is similar.

42. *Find the volume of the solid bounded by the two paraboloids* $z = x^2 + 2y^2$ *and* $z = 12 - 2x^2 - y^2$.

First Solution: First we find the intersection of the two paraboloids. if $x^2 + 2y^2 = z = 12 - 2x^2 - y^2$, then $12 = 3x^2 + 3y^2$ so $x^2 + y^2 = 4$. The interesection is a wobbly curve above this circle of radius 2. Let R be the interior of the circle. The volume is

$$\int\int_R [(12 - 2x^2 - y^2) - (x^2 + 2y^2)]\, dA = \int\int_R [12 - 3x^2 - 3y^2]\, dA. \tag{1}$$

From this there are different ways of proceeding. We can set this up as an iterated integral:

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 3x^2 - 3y^2)\, dy\, dx.$$

The inner integral is (with $a = \sqrt{4 - x^2}$)

$$3 \int_{-a}^{a} (a^2 - y^2)\, dy = 4a^3 = 4(4 - x^2)^{3/2},$$

so we get

$$4 \int_{-2}^{2} (4 - x^2)^{3/2}\, dx.$$

To solve this make the substitution $x = 2\sin(\theta)$, so $dx = 2\cos(\theta)$, $(4 - x^2)^{3/2} = 8\cos(\theta)$. As θ goes from $-\pi/2$ to $\pi/2$, x goes from -2 to 2, so the integral becomes

$$64 \int_{-\pi/2}^{\pi/2} \cos^4(\theta)\, d\theta,$$

and integral we've seen in class. Use

$$\cos^2(\theta) = \tfrac{1}{2}(1 + \cos(2\theta)). \tag{2}$$

Squaring this standard identity, then using it again to rewrite $\cos^2(2\theta)$ we get

$$\cos^4(\theta) = \tfrac{1}{4}(1 + 2\cos(2\theta) + \cos^2(2\theta)) = \tfrac{1}{4}(1 + 2\cos(2\theta) + \tfrac{1}{2}(1 + \cos(4\theta)))$$

or

$$\cos^4(\theta) = \tfrac{3}{8} + \tfrac{1}{2}\cos(2\theta) + \tfrac{1}{8}\cos(4\theta).$$

So

$$64 \int_{-\pi/2}^{\pi/2} \cos^4(\theta)\, d\theta, = [24\theta + 16\sin(2\theta) + 2\sin(4\theta)]_{\theta=-\pi/2}^{\theta=\pi/2}.$$

The sin's do not contribute since $\sin(2\theta) = \sin(4\theta) = 0$ when $\theta = \pm\pi/2$, and we get just 24π as our answer.

Second Solution: Although polar coordinates are not yet introduced in the text, we can get an answer more easily using them. Starting with equation (1), in polar coordinates $12 - 3x^2 - 3y^2 = 12 - 3r^2$, and $dA = r\,dr\,d\theta$. The integral equals

$$\int_0^{2\pi} \int_0^2 (12 - 3r^2)\, r\, dr\, d\theta = 2\pi \int_0^2 (12 - 3r^2)\, r\, dr.$$

Make the variable change $u = r^2$, $du = 2r\,dr$. When $0 < t < 2$, $0 < u < 4$, so we get

$$\pi \int_0^4 (12 - 3u)\, du = 24\pi.$$

Section 14.4.

4. *Compute the area of one loop of the rose* $r = 2\cos(2\theta)$.

See the book for a figure.

Solution. For one loop, θ will go from $-\pi/4$ to $\pi/4$ so we obtain

$$\int_{-\pi/4}^{\pi/4} \int_0^{2\cos(2\theta)} r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} 2\, \cos^2(2t)\, d\theta.$$

For this we use the identity

$$2\cos^2(2\theta) = 1 + \cos(4\theta),$$

which is essentially (2). We get

$$\int_{-\pi/4}^{\pi/4} (1 + \cos(4\theta)\, d\theta = [\theta + \tfrac{1}{4}\sin(4\theta)]_{\theta=-\pi/4}^{\theta=\pi/4} = \pi/2.$$

13. *Use polar coordinates to compute the integral*

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{1 + x^2 + y^2}\, dx\, dy.$$

Solution. The domain of integration is a quarter circle, and the integrand is $1/(1 + r^2)$. So the integral is:

$$\int_0^{\pi/2} \int_0^1 \frac{1}{1 + r^2}\, r\, dr\, d\theta = \frac{\pi}{2} \int_0^1 \frac{1}{1 + r^2}\, r\, dr.$$

To do this integral, make the substitution $u = r^2$, $du = 2\, r\, dr$. We get

$$\frac{\pi}{4} \int_0^1 \frac{1}{1 + u}\, du = \frac{\pi}{4}[\log|1 + u|]_{u=0}^{u=1} = \frac{\pi}{4}\log(2).$$

16. *Use polar coordinates to compute the integral*

$$\int_0^1 \int_x^1 x^2 \, dy \, dx.$$

Solution. In this case the solution using polar coordinates is *harder* because it is straightforward to do the integral directly:

$$\int_0^1 [x^2 y]_{y=x}^{y=1} \, dx = \int_0^1 (x^2 - x^3) \, dx = \frac{1}{12}.$$

But we are told to use polar coordinates so we'd better oblige. The domain of integration is the triangle with vertices (0,0),(1,1) and (0,1). Clearly θ can go from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, and r can go from 0 until we intersect the line $y = 1$. Since $y = r\sin(\theta)$, this means the domain of integration is

$$\int_{\pi/4}^{\pi/2} \int_0^{1/\sin(\theta)} \cdots \, dr \, d\theta.$$

We have $x^2 = r^2 \cos^2(\theta)$ and $dx\,dy = r\,dr\,d\theta$ so the actual integral is

$$\int_{\pi/4}^{\pi/2} \int_0^{1/\sin(\theta)} r^2 \cos^2(\theta) \, r \, dr \, d\theta = \int_{\pi/4}^{\pi/2} [r^4/4]_{r=0}^{r=1/\sin(\theta)} \cos^2(\theta) \, r \, dr \, d\theta =$$

$$\frac{1}{4} \int_{\pi/4}^{\pi/2} \frac{\cos^2(\theta)}{\sin^4(\theta)} \, d\theta.$$

Remember that the cotangent $\cot(\theta)$ equals $\cos(\theta)/\sin(\theta)$ and that the cosecant $\csc(\theta)$ (also denoted $\operatorname{cosec}(\theta)$) is $1/\sin(\theta)$, so we can write this

$$\frac{1}{4} \int_{\pi/4}^{\pi/2} \csc^2(\theta) \cot(\theta) \, d\theta.$$

Then (see the table of integrals in the back of the book, formula 9 there)

$$\int \csc^2(\theta) \, d\theta = -\cot(\theta) + C, \qquad \frac{d}{d\theta} \cot(\theta) = -\csc^2(\theta).$$

This means that we can use the substitution $u = \cot(\theta)$, $du = -csc^2(\theta)\,d\theta$. When $\theta = \pi/4$, $\cot(\theta) = 1$ and when $u = \pi/2$, $\cot(\theta) = 0$. Thus the integral is

$$\frac{1}{4} \int_1^0 u^2(-du) = \frac{1}{4} \int_0^1 u^2 \, du = \frac{1}{12}.$$

29. *Compute the volume of the "ice-cream cone" bounded by the sphere* $x^2 + y^2 + z^2 = a^2$ *and the cone* $z = \sqrt{x^2 + y^2}$.

Solution. We need to find the intersection. When both equations are tru,

$$a^2 - x^2 - y^2 = z^2 = x^2 + y^2$$

so $x^2 + y^2 = \frac{1}{2}a^2$, so $r = \sqrt{x^2+y^2} = \frac{1}{\sqrt{2}}a$. Using polar coordinates, the volume is

$$\int_0^{2\pi} \int_0^{a/\sqrt{2}} (\sqrt{a^2 - r^2} - r) r \, dr \, d\theta = 2\pi \int_0^{a/\sqrt{2}} (\sqrt{a^2 - r^2} - r) r \, dr.$$

The integral splits into two parts. The first is

$$2\pi \int_0^{a/\sqrt{2}} \sqrt{a^2 - r^2} \, r \, dr = \frac{1}{6}(4 - \sqrt{2}) a^3 \pi,$$

as may be shown using the substitution $u = r^2$. The second is

$$-2\pi \int_0^{a/\sqrt{2}} r^2 \, dr = \frac{1}{6}\sqrt{2}\, a^3 \pi.$$

Combining these and simplifying gives the answer $\frac{1}{3}(2 - \sqrt{2})\pi a^3$.

34. *Use the method of Example 5 to show that*

$$\int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2} dx \, dy = \frac{\pi}{4}.$$

Solution Actually the method of example 5 uses the beautiful "squaring" trick that is not involved in the solution of this problem. But here goes. The integral is over first quadrant, so it is

$$\int_0^{\pi/2} \int_0^\infty \frac{1}{(1 + r^2)^2} r \, dr \, d\theta = \frac{\pi}{2} \int_0^\infty \frac{1}{(1 + r^2)^2} r \, dr$$

The integral can be done using the substitution $u = r^2$ or $u = 1 - r^2$. We get

$$\frac{\pi}{2} \int_0^{2\pi} \left[\frac{-1}{2(1 + r^2)} \right]_{r=0}^{r=\infty}.$$

Now $\frac{-1}{2(1+r^2)} \to 0$ as $r \to \infty$, so this is just $\pi/4$.