Coulomb Force and Potential

by Daniel Bump for Math 52

We consider first the gravitational field and potential from a unit mass at the origin. I will assume that the gravitational constant $G = 1$. A particle of m at a point $\mathbf{x}$ will experience a force of magitude $m\,|\mathbf{x}|^{-2}$ pointing towards the attractor at the origin. This is the "inverse square law." The force is therefore a vector $-m\mathbf{x}/|\mathbf{x}|^3$. In other words, $m\mathbf{G}_0$, where $\mathbf{G}_0$ is the gravitational field due to this point mass at the origin:

$$\mathbf{G}_0(\mathbf{x}) = \frac{-\mathbf{x}}{|\mathbf{x}|^3} = \left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right),$$

where $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. It is easy to check that this is the gradient of:

$$\phi_0(\mathbf{x}) = \frac{1}{|\mathbf{x}|} = \frac{1}{\sqrt{x^2+y^2+z^2}}.$$

The gravitational field $\mathbf{G}_\mathbf{a}$ and potential $\phi_\mathbf{a}$ due to a point mass concentrated at $\mathbf{a}$ are obtained by translation. The gravitational field

$$\mathbf{G}_\mathbf{a}(\mathbf{x}) = \frac{-(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|^3} = \nabla \phi_\mathbf{a}(\mathbf{x})$$

where

$$\phi_\mathbf{a}(\mathbf{x}) = \frac{1}{|\mathbf{x}-\mathbf{a}|}.$$

Next consider the gravitational field and potential arising from a mass distributed in space. Let $\mu(\mathbf{a})$ denote the mass density near $\mathbf{a}$. Evidently the gravitational field should be obtained by averaging the gravitational fields $\mathbf{G}_\mathbf{a}$ due to point masses with the weight function $\mu(\mathbf{a})$:

$$\mathbf{G}_\mu(\mathbf{x}) = \int_{\mathbb{R}^3} \mu(\mathbf{a})\,\mathbf{G}_\mathbf{a}(\mathbf{x})\,dV_\mathbf{a} = \nabla \int_{\mathbb{R}^3} \mu(\mathbf{a})\,\phi_a(\mathbf{x})\,dV_\mathbf{a}. \tag{1}$$

This is an improper integral since ϕ_a blows up at $\mathbf{a} = 0$, but it may be seen to be convergent. The last equation follows immediately from $G_\mathbf{a} = \nabla\phi_\mathbf{a}$ by differentiation under the integral sign.

Since the gravitational field is a gradient, we have $\mathbf{curl}(\mathbf{G}_\mathbf{a}) = 0$. We also have $\operatorname{div}(\mathbf{G}_\mathbf{a}) = 0$ (except at $\mathbf{a}$), but this must be checked by a calculation. Since $\mathbf{G}_\mathbf{a}$ is obtained from $\mathbf{G}_0$ by a translation, it is sufficient to show that $\mathrm{div}(\mathbf{G}_0) = 0$ (except at the origin), and $\mathrm{div}(\mathbf{G}_0)(\mathbf{x}) =$

$$\frac{\partial}{\partial x}\left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right)+\frac{\partial}{\partial y}\left(\frac{-y}{(x^2+y^2+z^2)^{3/2}}\right)+\frac{\partial}{\partial z}\left(\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right)=$$

$$\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}+\frac{-x^2+2y^2-z^2}{(x^2+y^2+z^2)^{5/2}}+\frac{-x^2-y^2+2z^2}{(x^2+y^2+z^2)^{5/2}}=0.$$

Now let D be a domain in $\mathbb{R}^3$, with boundary ∂D. (**Note:** the boundary of a domain D is often denoted ∂D in topology. There is no partial derivative implied by this notation.) If $\mathbf{a} \notin D$, then by Gauss' Divergence Theorem the flux

$$\int_{\partial D} \mathbf{G_a} \cdot d\mathbf{S} = \int_D \mathrm{div}(\mathbf{G_a})\, dV = \int_D 0\, dV = 0.$$

This argument breaks down if $\mathbf{a} \in D$, but we can still prove without further calculation that there exists a constant c such that

$$\int_{\partial D} \mathbf{G_a} \cdot d\mathbf{S} = \begin{cases} c & \text{if } \mathbf{a} \in D; \\ 0 & \text{if } \mathbf{a} \notin D. \end{cases} \tag{2}$$

The second case we have already dealt with. We must show that if D_1 and D_2 are two domains such that $\mathbf{a} \in D_1$ and $\mathbf{a} \in D_2$, then

$$\int_{\partial D_1} \mathbf{G_a} \cdot d\mathbf{S} = \int_{\partial D_2} \mathbf{G_a} \cdot d\mathbf{S}. \tag{3}$$

First let us consider the case where $D_2 \subset D_1$. Let D_0 be the set-theoretic difference $D_1 - D_2$. This is a hollow domain, whose boundary consists of $\partial D_1 \cup \partial D_2$. Then $\mathbf{a} \notin D_0$, so by Gauss' Divergence Theorem,

$$0 = \int_{D_0} \mathrm{div}(\mathbf{G_a})\, dV = \int_{\partial D_1} \mathbf{G_a}\, d\mathbf{S} - \int_{\partial D_2} \mathbf{G_a}\, d\mathbf{S}.$$

The minus sign at the last step comes from the fact that the normal vectors on ∂D_2 are pointing away from $\mathbf{a}$ in calculating the flux $\int_{\partial D_2} \mathbf{G_a}\, d\mathbf{S}$, but are pointing towards $\mathbf{a}$ in calculating the flux $\int_{\partial D_0} \mathbf{G_a}\, d\mathbf{S}$,. Thus we have proved (3) in the special case where $D_2 \subset D_1$. Now let us consider the general case, still assuming that $\mathbf{a} \in D_1$ and $\mathbf{a} \in D_2$. Even though it may not be that either $D_2 \subset D_1$ or $D_1 \subset D_2$, still we may find $D_3 \subset D_1 \cap D_2$ with $\mathbf{a} \in D_3$. Then by what we have already proven,

$$\int_{\partial D_1} \mathbf{G_a} \cdot d\mathbf{S} = \int_{\partial D_3} \mathbf{G_a} \cdot d\mathbf{S} = \int_{\partial D_2} \mathbf{G_a} \cdot d\mathbf{S}.$$

This proves (3), and hence (2).

It remains to evaluate the constant c. We may take $\mathbf{a}$ to be the origin, and D to be the ball of radius 1 with center at 0. At a point $\mathbf{x}$ on the sphere ∂D, the vector $\mathbf{G}_0(\mathbf{x})$ is the inward-pointing normal vector of length 1, whose inner product with the outward-pointing normal vector of length 1 is -1. Thus the flux over this sphere is just minus the surface area of the unit sphere D. We see that $c = -4\pi$. This proves that

$$\int_{\partial D} \mathbf{G}_{\mathbf{a}} \cdot d\mathbf{S} = \begin{cases} -4\pi & \text{if } \mathbf{a} \in D; \\ 0 & \text{if } \mathbf{a} \notin D. \end{cases} \tag{4}$$

This equation may be written

$$\int_{\partial D} \mathbf{G}_{\mathbf{a}} \cdot d\mathbf{S} = -4\pi\chi_D(\mathbf{a}),$$

where the *characteristic function* χ_D is the function which is 1 on D, zero on its complement.

Now let us consider the flux of $\mathbf{G}_\nu$, the gravitational field arising from a distribution μ of mass in space. We will prove that the gravitational flux

$$\int_{\partial D} \mathbf{G}_\mu \cdot d\mathbf{S} = -4\pi M, \tag{5}$$

where

$$M = \int_D \mu(\mathbf{a})\, dV_{\mathbf{a}}$$

is the mass inside D. Indeed, substituting the definition (1) of $\mathbf{G}_\mu$, we have

$$\int_{\partial D} \mathbf{G}_\mu \cdot d\mathbf{S} = \int_{\partial D} \int_{\mathbb{R}^3} \mu(\mathbf{a})\, \mathbf{G}_{\mathbf{a}}\, dV_{\mathbf{a}} \cdot d\mathbf{S},$$

and interchanging the order of integration, this equals

$$\int_{\mathbb{R}^3} \int_{\partial D} \mu(\mathbf{a})\, \mathbf{G}_{\mathbf{a}} \cdot d\mathbf{S}\, dV_{\mathbf{a}} = \int_{\mathbb{R}^3} \mu(\mathbf{a}) \int_{\partial D} \mathbf{G}_{\mathbf{a}} \cdot d\mathbf{S}\, dV_{\mathbf{a}}.$$

Now making use of (4), this equals

$$-4\pi \int_{\mathbb{R}^3} \mu(\mathbf{a})\chi_D(\mathbf{a})\, dV_{\mathbf{a}}.$$

Remembering the definition of the characteristic function χ_D, this equals

$$-4\pi \int_D \mu(\mathbf{a}), dV_{\mathbf{a}} = M,$$

whence (5).

By Gauss' Divergence Theorem, we may rewrite (5) in the form

$$\int_D \operatorname{div}(\mathbf{G}_\mu)\, dV = -4\pi \int_D \mu\, dV.$$

Since this is true for every domain D, we have therefore:

$$\operatorname{div}(\mathbf{G}_\mu) = -4\pi\mu. \tag{6}$$

The equations (5) and (6) contain the same information, expressed in an integral form—(5)—versus a differential form—(6).