MATH 109 SAMPLE MIDTERM

Wednesday, February 25, 2004

Name: ____________________________________

Numeric Student ID: ________________________

Instructor's Name: __________________________

I agree to abide by the terms of the honor code:

Signature: _________________________________

Instructions: Print your name, student ID number and instructor's name in the space provided. During the test you may not use notes, books or calculators. Read each question carefully and **show all your work**; full credit cannot be obtained without sufficient justification for your answer unless explicitly stated otherwise. Underline your final answer to each question. There are 5 questions. You have 50 minutes to do all the problems.

Question	Score	Maximum
1		15
2		10
3		15
4		10
5		10
Total		60

1. Provide a brief definition AND an example of the following concepts:

(a) An equivalence relation $\sim$ on a set X.

Solution:

An equivalence relation is a binary operation on a set X satisfying

(1) $x \sim x$ for all $x \in G$.

(2) If $x \sim y$ then $y \sim x$ for all $x, y \in G$.

(3) If $x \sim y$ and $y \sim z$, then $x \sim z$ for all $x, y, z \in G$.

An example is conjugation where the set X is any group G.

(b) A normal subgroup H in a group G.

Solution:

A normal subgroup H is a subgroup of G which contains complete sets of conjugacy classes. Equivalently, a subgroup H is normal in G if

$$ghg^{-1} \in H \qquad \text{for all } h \in H, g \in G$$

There are many examples. Such as those in (c) and (d). $\{e\}$ is always a normal, if uninteresting, subgroup. Any subgroup of index 2 is also normal. And $\langle r^2 \rangle$ is normal in D_4, for example.

(c) The center $Z(G)$ of a group G.

Solution:

The center $Z(G)$ is a normal subgroup defined as the following subset of G:

$$\{x \mid xy = yx \quad \text{for all } y \in G\}$$

For example, in D_n, the center is $\{e\}$ if n is odd and $\{e, r^{n/2}\}$ if n is even.

(d) The commutator subgroup $[G, G]$ of a group G.

Solution:

The commutator subgroup of G is also a normal subgroup of G defined as the subgroup generated by the following set of elements:

$$\langle \{xyx^{-1}y^{-1} \mid x, y \in G\} \rangle$$

and the quotient of G by $[G, G]$ is the largest abelian quotient. Examples are $[S_n, S_n] = A_n$, $[\mathbb{Z}, \mathbb{Z}] = e$, and $[D_4, D_4] = \langle r^2 \rangle$.

2. Let H be a subgroup of G. Do sets of the form

$$S = \{y^{-1}x \mid x, y \in G, y^{-1}x \in H\}$$

partition G? Why or why not?

Solution:

We know that, given any equivalence relation $\sim$ on a set X, the equivalence classes of X form a partition of X (a collection of disjoint subsets which contain all the elements of X). We want to argue that the above set can be interpreted as equivalence classes under a binary operation $\sim$ defined according to $x \sim y$ for $x, y \in G$ if $y^{-1}x \in H$. To claim this, we must show that, in fact, $\sim$ is an equivalence relation.

But $x \sim x$ for all x in G since $x^{-1}x = e \in H$. And, given any x and y in G, if $x \sim y$, then $y^{-1}x \in H$ which implies $(y^{-1}x)^{-1} = x^{-1}y \in H$, i.e. that $y \sim x$. Finally, given $x, y, z \in G$ with $x \sim y$ and $y \sim z$, then $y^{-1}x$ and $z^{-1}y$ are in H, so their product is in H, which implies that $z^{-1}y * y^{-1}x = z^{-1}x$ is in H, i.e. that $x \sim z$. This proves we have an equivalence relation with equivalence classes of the form S above

In fact, one can check that the equivalence classes formed under $\sim$ are just left cosets of the group G.

3. Separate the following groups into isomorphism classes. That is, determine which groups, if any, are isomorphic to each other in the following list:

$$\mathbb{Z}/16\mathbb{Z}, \qquad \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \qquad D_8, \qquad D_4 \times \mathbb{Z}/2\mathbb{Z}, \qquad A_4$$

Be sure to explain your reasoning.

Solution:

All groups have order 16 except for A_4 which has 12 elements (as a subgroup of index 2 in S_4). So clearly A_4 cannot be isomorphic to any of the others. Now for the other groups, $\mathbb{Z}/16\mathbb{Z}$ is cyclic with generator 1, while $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has all elements with order 8 or less, but is abelian since both groups in the product are abelian. However, D_8 and $D_4 \times \mathbb{Z}/2\mathbb{Z}$ are not abelian, so the only remaining possibility is that these last two groups are isomorphic. But D_8 has elements of order 8, while the order of all elements of $D_4 \times \mathbb{Z}/2\mathbb{Z}$ divides 4. Hence, none of the groups in this list are isomorphic.

4. (a) Prove or Disprove: The group G_n defined for any integer n by

$$G_n = S_n/A_n \times D_n/\langle r \rangle \times \mathbb{Z}/7\mathbb{Z}$$

is abelian for any choice of n. Here $\langle r \rangle$ denotes the subgroup generated by a rotation r such that $r^n = e$.

Solution:

A_n is normal in S_n since it has index 2 and all such subgroups are normal. (Or note that it contains complete conjugacy classes according to the conjugacy classes of S_n). Then the quotient group S_n/A_n is a group with two elements under coset multiplication. But there's only one group of order 2 up to isomorphism, namely $\mathbb{Z}/2\mathbb{Z}$, so $S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$ (or you can just show the isomorphism explicitly). By identical reasoning, $\langle r \rangle$ is a subgroup of index 2 in D_n, hence normal, hence the quotient group is also isomorphic to $\mathbb{Z}/2\mathbb{Z}$. This implies that

$$G_n \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}.$$

(b) Prove or Disprove that the group G_n defined in part (a) is cyclic.

Solution:

According to the above arguments, we now know

$$G_n \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}.$$

But the largest order of any element in this group (of size 28) is 14, so it isn't cyclic. You can site, for example, the fact that the direct product of groups under modular arithmetic is cyclic if and only if their moduli are relatively prime.

5. Prove that all groups of order $2p$ are either cyclic or dihedral. That is, show that any such group G is isomorphic to C_{2p}, the cyclic group with $2p$ elements, or D_p, the group of symmetries of a p-gon.

Solution:

By Cauchy's theorem, any group of order $2p$ has an element of order p, call it x, and an element of order 2, call it y. Then $\langle x \rangle$, the subgroup generated by x, has index 2 in the group, and hence it is normal. Moreover, the elements of G can be given by the list

$$\{x, xy, x^2, x^2y, x^3, x^3y, \ldots, x^{p-1}, x^{p-1}y\}$$

Let's investigate the order of xy. We know, by Lagrange's theorem, that $xy^{|G|} = xy^{2p} = e$, so the possible orders of xy are $2p, p, 2$, or 1. But xy is not the identity, since y is its own inverse. If the order of xy is $2p$ the group is cyclic. If the order of $xy = 2$ then $xyxy = e$ which implies $xy = y^{-1}x^{-1} = yx^{-1}$ and this is just the dihedral relation so we get an immediate isomorphism with the dihedral group. Finally, we must show that there is no such group with order of $xy = p$, hence our group must be cyclic or dihedral.

If $(xy)^p = e$ then the coset

$$\langle x \rangle = \langle x \rangle (xy)^p = (\langle x \rangle xy)^p \quad \text{by definition of coset multiplication}$$

But

$$(\langle x \rangle xy)^p = (\langle x \rangle y)^p = \langle x \rangle y^p = \langle x \rangle y \quad \text{if } p \text{ is odd}$$

which is a contradiction, since $\langle x \rangle$ and $\langle x \rangle y$ are the two distinct cosets of the quotient group. If p is not odd (that is $p = 2$) then it follows since groups of order 4 are either $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong D_2$ or $\mathbb{Z}/4\mathbb{Z}$. This is certainly a hard problem. Probably way too hard for the time allowed in an exam.