

MATH 109 SAMPLE MIDTERM

Wednesday, January 28, 2004

Name: _____

Numeric Student ID: _____

I agree to abide by the terms of the honor code:

Signature: _____

Instructions: Print your name, and student ID number in the space provided. During the test you may not use notes, books, or calculators. Read each question carefully. Try to be as precise as possible when writing your proofs. Correct answers without justification will receive little or no credit. There are 6 questions. You have 50 minutes to do all the problems.

| Question | Score | Maximum |
|----------|-------|---------|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| Total | | 60 |

1. Determine if the following sets G with the indicated operation form a group by checking the group axioms. If not, point out which of the group axioms fail.
 - (a) $G =$ set of all integers \mathbb{Z} , with $a * b = a - b$.
 - (b) $G =$ set of all integers \mathbb{Z} , with $a * b = a + b + ab$.
 - (c) $G =$ set of all rational numbers $\mathbb{Q} \neq -1$, with $a * b = a + b + ab$.
 - (d) $G =$ set of all integers modulo 47, $\mathbb{Z}/47\mathbb{Z}$, with $a * b = ab \pmod{47}$.

2. Prove that every cyclic group G is abelian.

3. (a) Express the following elements of S_8 , the symmetric group of order $8!$, as a product of transpositions.

i. $(1\ 4\ 3)$

ii. $(2\ 8)(6\ 5\ 4)(4\ 5)$

(b) Express the following elements of S_{11} , the symmetric group of order $11!$, as a product of disjoint cycles.

i. $(7\ 4\ 3\ 8)(6\ 5\ 3)(2\ 1)(1\ 4\ 2)$

ii. $(6\ 5)(5\ 7)(8\ 4)(3\ 1\ 2)(3\ 5\ 11)$

4. Suppose that for a group G ,

$$(ab)^3 = a^3b^3 \quad \text{and} \quad (ab)^5 = a^5b^5 \quad \text{for any } a, b \in G.$$

Prove that G is abelian.

5. (a) Prove that for any finite group G , given any element $a \in G$, there is a positive integer n depending on a such that $a^n = e$.

- (b) Use the result above to show that for a finite group G , there is a positive integer m such that $a^m = e$ for all $a \in G$.

6. Find a shuffle of a deck of 52 cards that requires 42 repeats to return the cards to their original order.