CUBIC TWISTS OF $GL(2)$ AUTOMORPHIC L-FUNCTIONS

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Abstract. Let $K = \mathbb{Q}(\sqrt{-3})$ and let π be a cuspidal automorphic representation of $GL(2, \mathbb{A}_K)$. Consider the family of twisted L-functions $L(s, \pi \otimes \chi)$ where χ ranges over the cubic Hecke characters of K. In this paper the mean value of this family of L-functions is computed; the result is consistent with the generalized Lindelöf hypothesis. From this mean value result a nonvanishing theorem is established: for given s there are infinitely many cubic twists such that the L-value at s is nonzero. At the center of the critical strip the number of such characters of norm less than X is $\gg X^{1/6-\epsilon}$. These results are obtained by introducing and studying three different families of weighted double Dirichlet series. These series are related by functional equations, some of which are obtained through the study of higher metaplectic Eisenstein series and the Hasse-Davenport relation. The authors establish the continuation of such series and then obtain their main result by Tauberian methods.

0. Introduction. Let K be a number field containing n n^{th}-roots of unity and let π be an automorphic representation of $GL(r, \mathbb{A}_K)$. Denote by $L(s, \pi)$ the standard L-function associated to π normalized to have a functional equation under $s \to 1-s$. Let χ be a Hecke character for K and denote by $L(s, \pi, \chi)$ the twist of $L(s, \pi)$ by χ. The arithmetic and analytic properties of $L(s, \pi, \chi)$ for characters χ of fixed order n

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as the conductor of χ varies are of fundamental interest. For example, when $n = 2$ and $r \leq 3$ non-vanishing and mean value results have been obtained for $L(s, \pi, \chi)$ at $s = 1/2$ and in fact for any point s with $\Re s \geq 1/2$ (see [BFH3]). Arithmetic applications may then be obtained since, for example, the central values of $GL(2)$ L-functions attached to holomorphic modular forms of weight 2 are related to arithmetic quantities by means of the Theorem of Wiles and the Birch-Swinnerton-Dyer conjecture. See [GH,GHP,BFH1] for such applications. These results also play a role in Taylor's proof of the existence of a compatible system of l-adic Galois representations associated to an algebraic cuspidal automorphic representation of $GL(2)$ over an imaginary quadratic field; see [HST].

For $d \in K^\times$ let $||d||$ denote the absolute norm of d and let $\chi_d^{(n)}$ denote the Hecke character associated by class field theory to the extension $K(\sqrt[n]{d})/K$. Then these mean-value results have the rough form

$$\sum L(s, \pi, \chi_d^{(2)})\, W\left(\frac{||d||}{X}\right) \sim c(s, \pi)\, X, \tag{0.1}$$

where $W(||d||/X)$ is a suitable weighting function and $c(s, \pi)$ is a simple multiple of $L(2s, \pi, \vee^2)$, the symmetric square L-series of π at the argument $2s$. If π is taken to be the automorphic representation attached to a minimal parabolic Eisenstein series, then (0.1) becomes an r^{th} moment formula for $r = 1, 2, 3$. Though the conjectural generalization of (0.1) to arbitrary r and related conjectures about double Dirichlet series are straightforward to describe (see [BFH2, DGH,CFKRS]), full results—for example at $s = 1/2$—are not known, and in fact the methods of proof described in [BFH2-3] break down for $r \geq 4$. See [BFH2] for details and [CFH] for some results for arbitrary r for s inside the critical strip but sufficiently far from its center.

The situation for higher twists ($n > 2$) is still more complicated, and for arbitrary rank we do not even have the conjectures of [BFH2, DGH,CFKRS]. However when $r = 1$, analogous results have been obtained [FHL]. These have the form

$$\sum L(s, \pi, \chi_d^{(n)})\, W\left(\frac{||d||}{X}\right) \sim c^{(n)}(s, \pi)\, X. \tag{0.2}$$

Here as $r = 1$ π is simply a fixed Hecke character. The function $c^{(n)}(s, \pi)$ is essentially $L(ns, \pi^n)$. A similar result has been obtained in the case $r = 1, n = 3$ by Luo [L].

The object of this paper is to provide the first generalization of the formulas (0.1) and (0.2) to the case $n = 3$ and $r = 2$, i.e. to cubic twists of a $GL(2)$ automorphic representation π. For simplicity we will work over $K = \mathbb{Q}(\sqrt{-3})$, but our methods generalize to an arbitrary field containing the cube roots of 1. However our approach combines in an essential way multiple Dirichlet series methods with the Hasse-Davenport relation and with facts concerning metaplectic Eisenstein series and their residues. It is not apparent that the intricate blend which we give here generalizes to higher order twists or to higher degree L-functions.

To state our main result we first establish a bit of notation. Given $d \in \mathcal{O}_K$, $d \equiv 1 \bmod 3$, factor $d = d_1 d_2^2 d_3^3$ with each $d_i \equiv 1 \bmod 3$, d_1 square-free, $d_1 d_2^2$ cube-free. Let $P(s; d)$ denote the Dirichlet polynomial defined in (2.10). $P(s; d)$ depends on π but we suppress this from the notation. It is a complicated object, but has the properties that $P(s; d) = 1$ if $d_3 = 1$ and that, for fixed d_1, d_2, the sum

$$\sum_{d_3 \equiv 1 \mod 3} \frac{P(s; d_1 d_2^2 d_3^3)}{||d_3||^{3w}} \tag{0.3}$$

converges absolutely for $\Re w > 1/2$ and $\Re s \geq 1/2$. For a finite set of primes S, let $L_S(s, \pi, \chi^{(3)}_{d_1 d_2^2})$ denote the the L-series of π, twisted by $\chi^{(3)}_{d_1 d_2^2}$, and with Euler factors corresponding to the primes in S removed. Then we prove

Theorem 0.1. *Let $\pi = \otimes \pi_v$ be an automorphic representation of $GL(2, \mathbb{A}_K)$ such that $L(s, \pi, \chi)$ is entire for all Hecke characters χ such that $\chi^3 = 1$. Let S be a finite set of primes including the archimedean prime and the primes dividing $2, 3$ and the level of π. Then, for any sufficiently large positive integer k, the asymptotic formula*

$$\sum_{||d|| < X} L_S(s, \pi, \chi^{(3)}_{d_1 d_2^2}) P(s; d) \left(1 - \frac{||d||}{X}\right)^k \sim \frac{1}{k+1} c^{(3)}(s, \pi) X$$

holds for any s with $\Re s \geq 1/2$. The constant $c^{(3)}(s, \pi)$ is non-zero, and is given by

$$c^{(3)}(s, \pi) = \\ c_S L_S(3s, \pi, \vee^3) \zeta_S(6s) \zeta_S(6s+1)^{-1} \prod_{p \notin S} (1 - \gamma_p^3 ||p||^{-3s-1})(1 - \delta_p^3 ||p||^{-3s-1}),$$

where ζ_S denotes the Dedekind zeta function of K with the Euler factors at the places in S removed, γ_p, δ_p are the Satake parameters of the representation π_p, and c_S, defined in (4.2), is non-zero.

An immediate consequence of this, the convergence of (0.3), and the usual convexity bound for $L(1/2, \pi, \chi^{(3)}_{d_1 d_2^2})$ is

Corollary 0.2. *Let π be as in Theorem 0.1. Then there exist infinitely many cube-free d such that $L(1/2, \pi, \chi^{(3)}_{d}) \neq 0$. More precisely, let $N(X)$ denote the number of such d with $||d|| \ll X$. Then for any $\epsilon > 0$, $N(X) \gg X^{1/6-\epsilon}$.*

Our approach is via the method of multiple Dirichlet series. We define the multiple Dirichlet series

$$Z_1(s,w) = \sum_{d \equiv 1 \mod 3, (d,S)=1} \frac{L_S(s, \pi, \chi^{(3)}_{d_1 d_2^2}) P(s;d)}{||d||^w}.$$

Here the sum is over all $d \in \mathcal{O}_K$ with $d \equiv 1 \mod 3$ and $\mathrm{ord}_v(d) = 0$ for all finite $v \in S$, and the function $P(s;d)$ is the Dirichlet polynomial defined in (2.10) and mentioned above. This series converges absolutely for $\Re(s), \Re(w) > 1$. Our main result is the continuation of this function to a larger region; then Theorem 0.1 follows by standard Tauberian methods. Let

$$Z^*(s,w) = Z_1(s,w)\, \zeta_S(6s+6w-5)\, \zeta_S(12s+6w-8) \\ \times \prod_{p \notin S} (1-\gamma_p^3 ||p||^{2-3s-3w})^{-1} (1-\delta_p^3 ||p||^{2-3s-3w})^{-1},$$

where γ_p, δ_p are as above. We show that $Z^*(s,w)$ has a meromorphic continuation to the half plane $\Re(s+w) > 1/2$ and is analytic in this region except for polar lines at $w=1$, $w=0$, $w=5/3-2s$, $w=3/2-2s$, $w=4/3-2s$, $w=7/6-s$, $w=1-s$, $w=5/6-s$. Theorem 0.1 follows after computing the residue at $w=1$. We also show (Proposition 4.1) that the residue at $w=1$ satisfies

$$\mathrm{Res}_{w=1}\, Z^*(s,w) = c_S\, L_S(3s, \pi, \vee^3)\, \zeta_S(6s)\, \zeta_S(12s-2)$$

and is an analytic function of s for $\Re s > -1/2$, except possibly at the points $s = 1/3, 1/4, 1/6, 0$, which require a more detailed analysis. The properties of the symmetric cube L-series have been completely described in [KS].

In the remainder of this introduction we will sketch the ideas behind the meromorphic continuation of $Z^*(s,w)$. We will also explain why the polynomials $P(s;d)$

have been introduced. To do so, let us briefly recall the continuation of a similar double Dirichlet series

$$\sum \frac{L_S(s,\pi,\chi_d^{(2)})\, P_2(s;d)}{||d||^2} \tag{0.4}$$

corresponding to quadratic twists. For $r = 1, 2, 3$ such a double Dirichlet series may be obtained as an integral transform of a metaplectic Eisenstein series (see [BFH2] and the references there). After a great many technical difficulties, this observation can be used to establish the meromorphic continuation of (0.4). However, there is in fact a better way to study this object. This method was developed by Bump, Friedberg and Hoffstein (in the late 1990's) and explained in [BFH3].

The method of [BFH3] is based on two observations. First, requiring the double Dirichlet series (0.4) to also be expressible as a sum of L-functions in w (compare (0.6) below) and to satisfy a certain group of functional equations necessitates the introduction of the weight factors $P_2(s;d)$. Moreover these requirements uniquely determine these factors for $r \leq 3$. Second, applying the family of functional equations and associated Phragmén-Lindelöf convexity bounds to the region of absolute convergence of such a double Dirichlet series leads to an analytic continuation to an overlapping set of regions whose convex hull is $\mathbb{C}^2$. By the convexity principle for functions of two complex variables [Hö,Theorem 2.5.10] the analytic continuation of (0.4) to all of $\mathbb{C}^2$ then follows. This approach is worked out in detail in [DGH] over $\mathbb{Q}$, in [BFH3] for $n = 2, r = 3$ over an arbitrary number field and in [FF1-2] for $r = 1, 2$ in the case of an arbitrary function field. The computations involved are not trivial, but they are considerably easier than those required by the study of integral transforms of metaplectic Eisenstein series.

We return to the discussion of the series $Z^*(s,w)$. In this paper, we extend the approach of [BFH3] to this case. Various asymmetries in the cubic case create significant obstacles which must be overcome. To do so, we introduce three different families of double Dirichlet series, one as described above and two additional ones involving Gauss sums. We then establish functional equations relating these three objects. Suppressing details, our work is described as follows.

Write $d = d_1 d_2^2 d_3^3$ as above. Ignoring bad primes such as those dividing the level of π and the infinite place, $L(s,\pi,\chi^{(3)}_{d_1 d_2^2})$ has a functional equation of the form

$$L(s,\pi,\chi^{(3)}_{d_1 d_2^2}) \to \epsilon_\pi \tau(\chi^{(3)}_{d_1 d_2^2})^2 L(1-s,\tilde{\pi},\overline{\chi}^{(3)}_{d_1 d_2^2})||d_1 d_2||^{1-2s}. \tag{0.5}$$

Here $\tilde{\pi}$ denotes the contragredient of π, ϵ_π (the central value of the usual epsilon-factor for π) has absolute value 1 and $\tau(\chi_d^{(3)})$ is the usual Gauss sum associated to

$\chi_d^{(3)}$, normalized to have absolute value 1. The crucial factor $||d_1 d_2||^{1-2s}$ arises as part of the epsilon-factor of the twisted L-function since $\pi \otimes \chi_d^{(3)}$ is ramified at the primes dividing $d_1 d_2$. This functional equation gives rise to a functional equation for the double Dirichlet series Z_1, reflecting $Z_1(s,w)$ into a second double Dirichlet series

$$Z_6(s,w) = \sum \frac{L_S(s, \tilde{\pi}, \overline{\chi}^{(3)}_{d_1 d_2^2})\, \tau(\chi^{(3)}_{d_1 d_2^2})^2\, P(1-s; d_1 d_2^2 d_3^3)\, ||d_2 d_3^3||^{1-2s}}{||d_1 d_2^2 d_3^3||^{w}}.$$

More precisely, the functional equation (0.5) induces a transformation relating $Z_1(s,w)$ to $Z_6(1-s, w+2s-1)$. (The exact transformation is somewhat complicated due to bad primes; see Proposition 2.3.)

Next we study the series $Z_6(s,w)$ itself. The appearance of $\tau(\chi^{(3)}_{d_1 d_2^2})^2$, the square of a cubic Gauss sum, introduces, via the Hasse-Davenport relation [DH], a conjugate 6^{th} order Gauss sum. We will show, using the properties of Eisenstein series on the 6-fold cover of $GL(2)$, that $Z_6(s,w)$ possesses a functional equation as $(s,w) \rightarrow (s+2w-1, 1-w)$, transforming into itself.

As noted above, there is yet a third double Dirichlet series related to these two. For we will also show (Proposition 2.5) that the order of summation in $Z_1(s,w)$ written as a doubly-indexed Dirichlet series can be interchanged, leading to an expression of the form

$$Z_1(s,w) = \sum \frac{L_S(w, \chi^{(3)}_{m_1 m_2^2})\, Q(w; m_1 m_2^2 m_3^3)}{||m_1 m_2^2 m_3^3||^{s}}, \tag{0.6}$$

where Q is once again a specific Dirichlet polynomial depending on π and the L-series on the right are Hecke L-series. Applying the functional equation in w for the Hecke L-series we are led to introduce the third double Dirichlet series

$$Z_3(s,w) = \sum \frac{L_S(w, \overline{\chi}^{(3)}_{m_1 m_2^2})\, \tau(\chi^{(3)}_{m_1 m_2^2})\, Q(1-w; m_1 m_2^2 m_3^3)\, ||m_2 m_3^3||^{1/2-w}}{||m_1 m_2^2 m_3^3||^{s}}.$$

The functional equation for the Hecke L-series induces a transformation relating $Z_1(s,w)$ to $Z_3(s+w-1/2, 1-w)$.

Once again, the series Z_3 may be studied using metaplectic Eisenstein series. Indeed, we show that this series is a sum of cubic twists of Rankin-Selberg convolutions of π with the theta function on the 3-fold cover of $GL(2)$. (Recall that this function is the residue of an Eisenstein series on the 3-fold cover of $GL(2)$; see Patterson [Pa].) From the meromorphic continuation of the twisted Rankin-Selberg convolutions we deduce a corresponding continuation for Z_3.

We now use convexity methods to obtain the continuation of these 3 functions. The functions $Z_1(s,w)$ and $Z_6(s,w)$ have overlapping regions of absolute convergence. If the functional equation interchanging $Z_1(s,w)$ and $Z_6(s,w)$ is used several times, the convexity principle for several complex variables applied to the union of translates of these regions implies an analytic continuation of $Z_1(s,w)$ and $Z_6(s,w)$ to the half plane $\Re(w+s) > 3/2$. The relations with $Z_3(s,w)$ then imply an analytic continuation to the half plane $\Re(w+s) > 1/2$, which is what we require for our applications.

It is worth noting that a further functional equation transforming $Z_3(s,w)$ into itself as $(s,w) \to (1-s, w+4s-2)$, can be proved. This then allows an analytic continuation of all three double Dirichlet series to $\mathbb{C}^2$. This also gives rise to a group of functional equations which is non-abelian and of order 384. In order to keep the present work to a reasonable length these computations are deferred to another paper.

It is interesting to ask if the results obtained in this paper could also be obtained by other means. For example, one could try integrating results of Patterson [Pa] on mean values of cubic Gauss sums with the approximate functional equation and generalizations of the large sieve inequality. Luo has explored this in [L]. When $r = 1$, i.e. in the case of Hecke L-series, Luo succeeds in obtaining mean values back to $\Re s = 1/2$. However when $r = 2$, the case of this paper, the technique seems to fail to reach $\Re s = 1/2$.

As mentioned above, in the quadratic twist case the double Dirichlet series for $r = 1, 2, 3$ can be identified, up to a finite number of places, with certain integral transforms of metaplectic Eisenstein series. However significant technical problems are introduced at the archimedean place by this approach. This is because integral transforms of the Whittaker functions associated to metaplectic forms are rarely easy to analyze. Even at finite places the computational complexity becomes daunting as the rank increases. In the case at hand, although there is no known way to construct the double Dirichlet series as a similar integral transform (or as a Rankin-Selberg convolution), there is a natural candidate attached to the cubic cover of G_2, and it is possible that our complicated formulas reflect in a certain sense combinatorial issues arising from that group. Since the method of [BFH3] allows one to find the weight factors $P(s;d)$ directly, however, these considerations do not enter into the calculations presented here.

One may also obtain a mean value result for the product of two Hecke L-functions in different variables when they are simultaneously twisted by cubic characters. This has been accomplished by the first-named author in his Brown University doctoral dissertation [Br].

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1. Preliminaries. For the remainder of this paper let $K = \mathbb{Q}(\sqrt{-3})$ and let π be an automorphic representation of $GL(2, \mathbb{A}_K)$. If N denotes the level of π and S is a finite collection of primes including all the primes dividing $6N$ and also the archimedean prime, let M equal 6^3 (not optimal) times the product of all the finite primes in S. We fix $n = 3$ and usually we write χ_d instead of $\chi_d^{(3)}$.

For $m \in K$ let $||m||$ denote the norm of m. Since every finite prime not dividing 3 is primitive, the L-series of π with the Euler factors corresponding to the primes in S removed is given by

$$L_S(s, \pi) = \sum_{m \equiv 1 \mod 3, (m,S)=1} \frac{c(m)}{||m||^s}.$$

Here $c(m)$ may be obtained as the m^{th} Fourier coefficient of an automorphic form in the space of π and the sum converges absolutely for $\Re s > 1$. Similarly, let ψ be a primitive cubic Hecke character with conductor dividing M. Then the L-series of π, twisted by χ_d and ψ and with the Euler factors corresponding to the primes in S removed, is given by

$$L_S(s, \pi, \chi_d\psi) = \sum_{m \equiv 1 \mod 3, (m,S)=1} \frac{c(m)\chi_d(m)\psi(m)}{||m||^s}.$$

Let us extend the notation at the unramified places to all finite places p by writing

$$L_p(s, \pi, \chi_d\psi) = (1 - \alpha_p \chi_d(p)\psi(p)||p||^{-s})^{-1}(1 - \beta_p \chi_d(p)\psi(p)||p||^{-s})^{-1}$$

if $p < \infty$; note that if $p|M$ then α_p, β_p may be 0. It is convenient to separate out the archimedean place ∞ and write $L_{\{\infty\}}(s, \pi)$ as an Euler product:

$$\begin{aligned} L_{\{\infty\}}(s, \pi, \chi_d\psi) = L_S(s, \pi, \chi_d\psi) \\ \times \prod_{p|M} (1 - \alpha_p \chi_d(p)\psi(p)||p||^{-s})^{-1}(1 - \beta_p \chi_d(p)\psi(p)||p||^{-s})^{-1}. \end{aligned}$$

This last series has an analytic continuation to all of $\mathbb{C}$ and satisfies a functional equation of the form

$$(1.1)\quad ||d_1^2 d_2^2 N_{\pi,\psi}||^{s/2}\Gamma_{\pi,\psi}(s)L_{\{\infty\}}(s,\pi,\chi_d\psi) = \Lambda(s,\pi,\chi_d\psi) = \epsilon_{\pi,\psi}\tau(\chi_d)^2\psi'_{\pi,\psi}(d)\Lambda(1-s,\tilde{\pi},\overline{\chi}_d\overline{\psi}).$$

Here $\tilde{\pi}$ denotes as above the contragredient of π and $\Gamma_{\pi,\psi}(s)$ is a product of gamma factors. As K is totally complex, the gamma factors will not depend on d. Also $\epsilon_{\pi,\psi}$ has absolute value 1, $\psi'_{\pi,\psi}$ is a character of conductor dividing M, and the level $N_{\pi,\psi}$ of $\pi\otimes\psi$ divides M^2N. Finally, $\tau(\chi_d)$ is the usual Gauss sum associated to χ_d, normalized to have absolute value 1.

The incomplete L-series $L_S(s,\pi,\chi_d\psi)$ also satisfies a functional equation. This involves a scattering matrix, and it is convenient to write it in vector form. Let Ψ denote the set of all primitive cubic characters ψ with conductors dividing M (including the trivial character) and let C denote the cardinality of Ψ. Let $\overrightarrow{\mathbf{L}}_S(s,\pi,\chi_d)$ denote the $C\times 1$ column vector whose entries are $L_S(s,\pi,\chi_d\psi^{(j)})$, where $\psi^{(j)}$ ranges over all characters in Ψ. Then the following is an immediate consequence of (1.1), Stirling's formula, and the analyticity of $L(s,\pi,\chi_d\psi)$.

Proposition 1.1. *Let $d = d_1d_2^2$, with d_1, d_2 square free, $(d_1,d_2)=1$, $d_1\equiv d_2\equiv 1$ mod 3. Then the vector $\overrightarrow{\mathbf{L}}_S(s,\pi,\chi_d)$ converges absolutely for $\Re s>1$ and continues to an entire function of order 1. Further, there exists a $C\times C$ matrix $\Phi_{\pi,M}(s,d)$ such that*

$$\overrightarrow{\mathbf{L}}_S(s,\pi,\chi_d)\prod_{p|M}(1-\tilde{\alpha}_p^3||p||^{-3+3s})(1-\tilde{\beta}_p^3||p||^{-3+3s}) = \tau(\chi_d)^2\Phi_{\pi,M}(s,d)\overrightarrow{\mathbf{L}}_S(1-s,\tilde{\pi},\overline{\chi}_d)||d_1d_2||^{1-2s}.$$

The entries of $\Phi_{\pi,M}(s,d)$, denoted $\Phi_{\pi,M}^{k,l}(s,d)$, are analytic functions in $\mathbb{C}$ except for possible poles corresponding to gamma factors in $\Gamma_{\pi,\psi}(s)$. These are canceled by trivial zeros of the L-series appearing in $\overrightarrow{\mathbf{L}}_S(1-s,\tilde{\pi},\overline{\chi}_d)$. For s in bounded vertical strips, i.e. for $s=\sigma+it$ with $K_1<\sigma<K_2$, the upper bound $\Phi_{\pi,M}^{k,l}(s,d)\ll |t|^A||M||^B$ holds for $|t|$ sufficiently large and A, B depending on K_1, K_2. The implied constant depends only on the eigenvalues of π.

There are three additional related objects whose analytic properties and functional equations we will require. The first of these is the ordinary Hecke L-series associated to χ and ψ. As before, there are the incomplete and complete versions (complete in the context of analytic number theory, i.e. corresponding to the complete Dirichlet series but not the archimedean factors),

$$L_S(w,\chi_m\psi) = \sum_{d\equiv 1 \mod 3, (d,S)=1} \frac{\chi_m(d)\psi(d)}{||d||^w}$$

and

$$L_{\{\infty\}}(w,\chi_m\psi) = L_S(w,\chi_m\psi)\prod_{p|M}(1-\chi_m(p)\psi(p)||p||^{-s})^{-1},$$

respectively. Let $m \equiv 1 \mod 3$, m cube-free and write $m = m_1m_2^2$, with $m_1, m_2 \equiv 1 \mod 3$, m_1, m_2 square-free. Both $L_{\{\infty\}}(w,\chi_m\psi)$ and $L_S(w,\chi_m\psi)$ are analytic except possibly at $w = 1$. They are also analytic at $w = 1$ unless ψ is trivial and $m = 1$, in which case there is a simple pole. The complete series $L_{\{\infty\}}(w,\chi_m\psi)$ satisfies a functional equation of the form

$$\begin{aligned} (1.2)\quad ||m_1m_2N_\psi||^{w/2}\Gamma_\psi(w)L_{\{\infty\}}(w,\chi_d\psi) &= \Lambda(w,\chi_d\psi) \\ &= \epsilon_\psi\tau(\chi_m)\psi'(m)\Lambda(1-s,\overline{\chi}_d\overline{\psi}). \end{aligned}$$

Here ϵ_ψ has absolute value 1, $\tau(\chi_m)$ is the usual Gauss sum associated to χ_m normalized to have absolute value 1, and ψ' is a primitive cubic character of conductor dividing M, depending on ψ.

A functional equation for the incomplete L-function may be deduced from (1.2), similarly to Proposition 1.1. Let $\overrightarrow{\mathbf{L}}_S(w,\chi_m)$ denote the $C\times 1$ column vector whose entries are the functions $L_S(w,\chi_m\psi^{(j)})$, where $\psi^{(j)}$ ranges over all characters in Ψ. Then

Proposition 1.2. *The function $w(w-1)\overrightarrow{\mathbf{L}}_S(w,\chi_m)$ converges absolutely for $\Re w > 1$ and extends to an entire function of order 1. There exists a $C\times C$ matrix $\Phi_M(w)$ such that*

$$\overrightarrow{\mathbf{L}}_S(w,\chi_m)\prod_{p|M}(1-||p||^{-3+3w}) = \tau(\chi_m)\Phi_M(w)\overrightarrow{\mathbf{L}}_S(1-w,\overline{\chi}_m)||m_1m_2||^{1/2-w}.$$

The entries $\Phi_M^{k,l}(w)$ of $\Phi_M(w)$ are analytic functions in $\mathbb{C}$ except for possible poles corresponding to gamma factors in $\Gamma_\psi(w)$. Except possibly at 0, these are canceled by trivial zeros of the L-series appearing in $\overrightarrow{\mathbf{L}}_S(1-w,\overline{\chi}_m)$. For w in bounded *vertical strips, i.e. for* $w = \sigma + it$ *with* $K_1 < \sigma < K_2$*, the upper bound* $\Phi_M^{k,l}(w) \ll |t|^A ||M||^B$ *holds for* $|t|$ *sufficiently large and* A, B *depending on* K_1, K_2*.*

The next objects we require are related to Eisenstein series on the 3-fold and 6-fold covers of $GL(2, \mathbb{A}_K)$. To define them we first need to establish some notation.

Let $d \in \mathcal{O}_K$ with $(d, 6) = 1$. Let $e(x) = e^{2\pi i \operatorname{Tr}(x/\sqrt{-3})}$. Then for $\ell = 3$ or 6 the normalized ℓ^{th} order Gauss sum modulo d with numerator m is defined by

$$G^{(\ell)}(m, d) = (\sqrt{||d||})^{-1} \sum_{r \mod d, (r,d)=1} \left(\tfrac{r}{d}\right)_\ell e(mr/d). \tag{1.3}$$

Note that d need not be cube-free in this definition. In the case that d is square-free this corresponds to the usual definition of the normalized Gauss sum, and for square-free d and $\ell = 3$, $G^{(3)}(1, d) = \tau(\chi_d^{(3)})$ and $|G^{(3)}(1, d)| = 1$.

Define a Dirichlet series built out of 6^{th} order Gauss sums as follows. Let

$$D_S(w, m) = \sum_{d \equiv 1 \mod 3, (d,S)=1} \frac{G^{(6)}(m, d)}{||d||^w} \tag{1.4}$$

and

$$D_S^*(w, m) = \zeta_S(6w - 2) D_S(w, m),$$

where ζ_S is again the Dedekind zeta function of K with the Euler factors in S removed. The function $D_S^*(w, m)$ converges for $\Re w > 1$ and may be realized as a Fourier coefficient of $E^{(6)}(z, w/2 + 1/4)$, an Eisenstein series on the 6-fold cover of $GL(2, \mathbb{A}_K)$. The analytic properties of $D_S^*(w, m)$ are inherited from $E^{(6)}(z, w/2 + 1/4)$. These properties were first investigated for general $n \geq 3$ by Kubota [Ku], who showed that $D_S^*(w, m)$ is analytic except for simple poles at $w = 2/3, 1/3$. The residue at $w = 2/3$ is given by

$$\operatorname{Res}_{w=2/3} D_S^*(w, m) = ||m||^{-1/6} \tau_6(m), \tag{1.5}$$

where $\tau_n(m)$ is, up to a constant multiple, the m^{th} Fourier coefficient of the theta function on the n-fold cover of $GL(2, \mathbb{A}_K)$ defined by

$$\theta^{(n)}(z) = \operatorname{Res}_{w=1/2+1/n} E^{(n)}(z, w/2 + 1/4). \tag{1.6}$$

See [Ho] for an exposition of the curious properties of $\tau_n(m)$ and $\theta^{(n)}(z)$ and also (1.7) below for the case $n = 3$.

It follows from the work of [KP] and [Ku] that $D^*_S(w,a)$ possesses a functional equation of $GL(1)$ type, with scattering matrix, similar to that of $L_S(w,\chi_m\psi)$. To state this, let $\Psi^{(6)}$ (of cardinality $C^{(6)}$) be the set of all primitive characters with conductor dividing M whose 6th powers are trivial. Then for each $\psi\in\Psi^{(6)}$ there is an associated integer $m_\psi\in\mathcal{O}_K$ such that for $d\in\mathcal{O}_K$, $d\equiv 1 \mod 3$, one has $\psi(d)=\left(\frac{m_\psi}{d}\right)_6$. Let $\overrightarrow{\mathbf{D}}^*_S(w,m)$ denote the $C^{(6)}\times 1$ column vector whose entries are the Dirichlet series $D^*_M(w,m_\psi m)$, where ψ ranges over all characters in $\Psi^{(6)}$. The functional equation for $\overrightarrow{\mathbf{D}}^*_S(w,m)$ has a form similar to Proposition 1.2, but without the Gauss sum. Stated precisely, we have

Proposition 1.3. *The function* $(w-2/3)(w-1/3)\overrightarrow{\mathbf{D}}^*_S(w,m)$ *converges absolutely for* $\Re w>1$. *In fact for* $\Re w>1+\epsilon$ *it satisfies* $\overrightarrow{\mathbf{D}}^*_S(w,m)\ll_\epsilon 1$ *with the implied constant independent of* M *and* ϵ. *It continues to an entire function of order 1 and there exists a* $C\times C$ *matrix* $\Phi^{(6)}_M(w)$ *such that*

$$\overrightarrow{\mathbf{D}}^*_S(w,m)\prod_{p|M}(1-||p||^{-n/2-1+nw})=\Phi^{(6)}_M(w)D^*_S(1-w,m)||m|^{1/2-w}.$$

The entries $\Phi^{(6)k,l}_M(w)$ *of* $\Phi^{(6)}_M(w)$ *are analytic functions in* $\mathbb{C}$ *except for possible poles cancelled by trivial zeros of* $D^*_S(1-w,m)$. *For* w *in bounded vertical strips, i.e. for* $w=\sigma+it$ *with* $K_1<\sigma<K_2$, *the upper bound* $\Phi^{(6)k,l}_M(w)\ll|t|^A||M||^B$ *holds for* $|t|$ *sufficiently large and* A,B *depending on* K_1,K_2.

Let $\tau(m)$ denote the m^{th} Fourier coefficient of the cubic theta function $\theta^{(3)}(z)$. If we write $m=m_1m_2^2m_3^3$ with $m_1\equiv m_2\equiv m_3\equiv 1 \mod 3$, m_1,m_2 square free, and $(m_1,m_2)=1$, then $\tau(m)$ is given [Pa] by $\tau(m)=0$ if $m_2\neq 1$ and

$$\tau(m_1m_3^3)=||m_3||^{1/2}\overline{G^{(3)}(1,m_1)} \tag{1.7}$$

otherwise.

The last series to be discussed is the Rankin-Selberg convolution of the automorphic representation π with the conjugate of $\theta^{(3)}(z)$. The incomplete series has the form

$$L_S(s,\pi\otimes\overline{\theta^{(3)}})=\sum_{m\equiv 1 \mod 3,(m,S)=1}\frac{c(m)\,\overline{\tau(m)}}{||m||^s}.$$

As we are working on $GL(2)$, the construction of a series which agrees with this one (except at a finite number of places) via a Rankin-Selberg integral is well understood even though $\theta^{(3)}(z)$ is (genuine) metaplectic and π is not. In fact the construction is an exact analog of the Rankin-Selberg convolution of π with the quadratic theta function which was used by Shimura [Sh] to construct the symmetric square L-series of π.

We will need some information about the analytic properties of the twisted convolution

$$L_S(s,\pi\otimes\overline{\theta^{(3)}}\otimes\overline{\chi}_d\psi)=\sum_{m\equiv 1 \mod 3,(m,S)=1}\frac{c(m)\,\overline{\tau(m)}\,\overline{\chi}_d(m)\,\psi(m)}{||m||^s},$$

where $d=d_1d_2^2$ as in Proposition 1.1 above. Each such function may also be expressed as a Rankin-Selberg integral. Let $\overrightarrow{\mathbf{L}}^*_S(s,\pi\otimes\theta^{(3)}\otimes\overline{\chi}_d)$ denote the $C\times 1$ column vector whose entries are

$$(s-1/3)(s-2/3)\zeta(6s-2)L_S(s,\pi\otimes\overline{\theta^{(3)}}\otimes\overline{\chi}_d\psi^{(j)}),$$

where $\psi^{(j)}$ ranges over all characters in Ψ. The factors $(s-1/3)(s-2/3)\zeta(6s-2)$ have been added to clear the denominator of the Rankin-Selberg convolution and to cancel possible poles at $s=1/3,2/3$. Then we have

Proposition 1.4. *The function* $\overrightarrow{\mathbf{L}}^*_S(s,\pi\otimes\overline{\theta^{(3)}}\otimes\overline{\chi}_d)$ *converges absolutely for* $\Re s>1$. *It continues to an entire function of order 1 and there exists a* $C\times C$ *matrix* $\Phi^\pi_M(s,d)$ *such that*

$$\overrightarrow{\mathbf{L}}^*_S(s,\pi\otimes\overline{\theta^{(3)}}\otimes\overline{\chi}_d)\prod_{p|M}(1-||p||^{6s-4})=\Phi^\pi_M(s,d)\,\overrightarrow{\mathbf{L}}^*_S(1-s,\pi\otimes\overline{\theta^{(3)}}\otimes\overline{\chi}_d).$$

The entries $\Phi^{\pi,k,l}_M(s,d)$ *of* $\Phi^\pi_M(w)$ *are analytic functions in* $\mathbb{C}$. *For* s *in bounded vertical strips, i.e. for* $s=\sigma+it$ *with* $K_1<\sigma<K_2$, *the upper bound* $\Phi^{\pi,k,l}_M(s)\ll(|t|||Md||)^A$ *holds for* $|t|$ *sufficiently large and* A *depending on* K_1,K_2.

Remark 1.5. The polynomial bound in $\underline{d}$ given above is all we require, and follows easily from the expression of $L_S(s,\pi\otimes\overline{\theta^{(3)}}\otimes\overline{\chi}_d\psi^{(j)})$ as a Rankin-Selberg integral. However, a more refined result can be proved with more effort:

$$\overrightarrow{\mathbf{L}}^*_S(s,\pi\otimes\overline{\theta^{(3)}}\otimes\overline{\chi}_d)\prod_{p|M}(1-||p||^{6s-4})=\Phi^\pi_M(s)\,\overrightarrow{\mathbf{L}}^*_S(1-s,\pi\otimes\overline{\theta^{(3)}}\otimes\overline{\chi}_d)||d_1d_2||^{2-4s},$$

where the scattering matrix $\Phi^\pi_M(s)$ is now independent of d.

2. Multiple Dirichlet series. Let π be as above. In this Section we shall introduce and study two of the three multiple Dirichlet series attached to π that were described in Section 0.

For $p \in \mathcal{O}_K$ prime, $p \equiv 1 \mod 3$, $(p, S) = 1$, let $\gamma(p), \delta(p)$ be the corresponding Satake parameters (we also write γ_p, δ_p on occasion), so $c(p) = \gamma(p) + \delta(p)$. Denote the corresponding parameters of the contragradient $\tilde{\pi}$ by $\tilde{\gamma}(p), \tilde{\delta}(p)$. It will be convenient to introduce the following notation. For $m = p_1^{a_1} p_2^{a_2} \cdots p_l^{a_l}$ we let

$$\gamma(m) = \gamma(p_1)^{a_1} \gamma(p_2)^{a_2} \cdots \gamma(p_l)^{a_l}, \quad \delta(m) = \delta(p_1)^{a_1} \delta(p_2)^{a_2} \cdots \delta(p_l)^{a_l}$$

and define $\tilde{\gamma}(m), \tilde{\delta}(m)$ similarly.

For $a, b, c \geq 0$, $a \geq b$, we define $R(p^a, p^b; p^c)$ as follows. For fixed a, b, choose $\epsilon \in \{0, 1, 2, 3, 4, 5\}$, $\delta \in \{0, 1, 2\}$ by $a + b \equiv \epsilon \mod 6$ and $a + b \equiv \delta \mod 3$.

If $0 \leq c \leq (a + b - \epsilon)/2$ and $c/3 \leq b$, or if $1 + (a + b - \epsilon)/2 \leq c \leq a + b - \delta$ and $c/3 \leq b \leq 2a - c - \delta - 1$, then for $c \equiv 0 \mod 3$

$$R(p^a, p^b; p^c) = p^{[(c + \min(b,c))/2]}. \tag{2.1}$$

Here $[x]$ denotes the greatest integer less than or equal to x.

If $1 + (a + b - \epsilon)/2 \leq c \leq a + b - \delta$ and $b \geq 2a - c - \delta$ then for $c \equiv 0 \mod 3$

$$R(p^a, p^b; p^c) = p^{(a+b+c-\delta)/3} \times \begin{cases} 1 & \text{if } \delta = 0, 2; \\ 2 & \text{if } \delta = 1. \end{cases} \tag{2.2}$$

If $1 + (a + b + \epsilon)/2 \leq c \leq a + b - 2$ and $b \geq 2a + 1 - c$ with $\delta = 0, c \equiv 1 \mod 3$, or if $(a + b + \epsilon)/2 \leq c \leq a + b$ and $b \geq 2a + 1 - c$ with $\delta = 2, c \equiv 2 \mod 3$, then

$$R(p^a, p^b; p^c) = p^{(a+b+c-1)/3}. \tag{2.3}$$

Extend the definition of $R(p^a, p^b; p^c)$ multiplicatively to $R(m_a, m_b; r)$ by

$$R(m_a m_a', m_b m_b'; r r') = R(m_a, m_b; r)\, R(m_a', m_b'; r')$$

for $(m_a m_b r, m_a' m_b' r') = 1$.

We are now ready to give a precise definition of one of the three Dirichlet series described in the Introduction. Recall that $\Psi^{(6)}$ denotes the set of all primitive characters ψ with conductor dividing M and such that $\psi^6 = 1$. Fix $\psi^{(i)}, \psi^{(j)} \in \Psi^{(6)}$. For $s, w \in \mathbb{C}$ with $\Re s > 1, \Re w > 1$ let

$$(2.4)\quad Z_6(s,w;\psi^{(i)},\psi^{(j)}) = \sum_{m\equiv 1 \mod 3, (m,S)=1} \frac{\sum_{r|m} F(m,r)\,\psi^{(i)}(m)\,D_S(w,m_{\psi^{(j)}}m^2/r^2)\,||r||^{-w}}{||m||^s}$$

where $D_S(w,m_{\psi^{(j)}}m^2/r^2)$ is as defined in (1.4) and

(2.5)

$$F(m,r) = \bar{\chi}_{r_1r_2^2}((m/r)_0)\overline{G^{(6)}(1,r_1)}G^{(6)}(1,r_2)\sum_{m=m_am_b} R(m_a,m_b;r)\tilde{\gamma}(m_a)\tilde{\delta}(m_b).$$

Here, for $r|m$ we have written $r = r_1r_2^2r_3^3$, with r_1, r_2 square-free, $(r_1,r_2)=1$ and $r_i \equiv 1(3), i = 1,2,3$. Also we let $(m/r)_0$ denote the cube free part of m/r. More generally for any $d \in \mathcal{O}_K$, $d \equiv 1 \bmod 3$ we will let d_0 denote the cube free part of d.

Let $\overrightarrow{\mathbf{Z}}_6(s,w)$ denote the length ${C^{(6)}}^2$ column vector whose entries are the functions $Z_6(s,w;\psi^{(i)},\psi^{(j)})$ and let

$$\overrightarrow{\mathbf{Z}}^*_6(s,w) = (w-2/3)(w-1/3)(w+2s-1)(w+2s-2)\zeta(6w-2)\,\overrightarrow{\mathbf{Z}}_6(s,w).$$

The following functional equation is now an immediate consequence of Proposition 1.3.

Proposition 2.1. *The function $\overrightarrow{\mathbf{Z}}^*_6(s,w)$ converges absolutely for $\Re s > 1, \Re w > 1$. It has an analytic continuation to the region R_1 consisting of points (s,w) for which $\Re s > 1$ for $\Re w > 1$; $\Re s > 2 - \Re w$ for $1 \geq \Re w \geq 0$; and $\Re s > 2 - 2\Re w$ for $0 > \Re w$. Inside R_1 it satisfies the functional equation*

$$\overrightarrow{\mathbf{Z}}^*_6(s,w)\prod_{p|M}(1-||p||^{-4+6w}) = \Phi_{6,6}(w)\,\overrightarrow{\mathbf{Z}}^*_6(s+2w-1,1-w).$$

Here $\Phi_{6,6}(w)$ is a ${C^{(6)}}^2 \times {C^{(6)}}^2$ matrix. The entries $\Phi^{k,l}_{6,6}(w)$ are analytic functions in $\mathbb{C}$ except for possible poles cancelled by trivial zeros of $D^*_S(1-w,m)$. For w in bounded vertical strips, i.e. for $w = \sigma + it$ with $K_1 < \sigma < K_2$, the upper bound $\Phi^{k,l}_{6,6}(w) \ll |t|^A||M||^B$ holds for $|t|$ sufficiently large and A, B depending on K_1, K_2.

Proof: Recall that for $\Re w > 1+\epsilon$, $\overrightarrow{\mathbf{D}}^*_S(w,m) \ll_\epsilon 1$. The definition of $R(p^a,p^b;p^c)$ in (2.1)–(2.3) is such that for $\Re w > 1+\epsilon$

$$\sum_{r|m} F(m,r)||r||^{-\Re w} \ll 1.$$

Thus $\overrightarrow{\mathbf{Z}}^*_6(s,w)$ converges absolutely for $\Re s > 1, \Re w > 1$. By Proposition 1.3 $\overrightarrow{\mathbf{D}}^*_S(w,m^2/r^2)||r||^{-w}$ reflects into $\overrightarrow{\mathbf{D}}^*_S(1-w,m^2/r^2)||r||^{-(1-w)}||m||^{1-2w}$. This, together with the Phragmén-Lindelöf principle suffices to establish the Proposition. ■

We turn next to the precise definition of the multiple Dirichlet series of most immediate interest to us. This requires a definition of the polynomials $P(s;d)$ which appear in Section 0. To give these, we define a class of functions $A(p^i,p^j;p^\alpha)$. Writing $\alpha=\alpha_1+2\alpha_2+3\alpha_3$, with $\alpha_1+2\alpha_2=0,1,2$, the following gives a description of the $A(p^i,p^j;p^\alpha)$ in terms of the $R(p^a,p^b;p^c)$ in the three cases (i) $\alpha_1=1,\alpha_2=0$, (ii) $\alpha_1=0,\alpha_2=1$, and (iii) $\alpha_1=\alpha_2=0$. For $k_1,k_2,\geq 0$, set

$$\text{(2.6)} \quad C(p^{k_1},p^{k_2};p^\alpha) = \sum_{0\leq l\leq \min(\alpha,k_1+k_2)} R(p^{k_1},p^{k_2};p^l)\overline{G^{(6)}(1,p^{l_1})}G^{(6)}(1,p^{l_2})\overline{G^{(6)}(p^{2(k_1+k_2-l)},p^{\alpha-l})}.$$

Here, just as with α, we have written $l=l_1+2l_2+3l_3$ with $l_1+2l_2=0,1,2$. Then for $i,j\leq \alpha_2+3\alpha_3$, $i+j\equiv 0 \mod 3$, we define

$$\text{(2.7)} \quad A(p^i,p^j;p^\alpha) = p^{i+j-\alpha} \times \begin{cases} G^{(6)}(1,p)C(p^{3\alpha_3-i},p^{3\alpha_3-j};p^{1+3\alpha_3}) & \text{if } \alpha_1=1,\alpha_2=0; \\ \overline{G^{(6)}(1,p)}C(p^{1+3\alpha_3-i},p^{1+3\alpha_3-j};p^{2+3\alpha_3}) & \text{if } \alpha_1=0,\alpha_2=1. \end{cases}$$

For $i,j\leq 3\alpha_3$

$$\text{(2.8)} \quad \begin{aligned} A(p^i,p^j;p^{3\alpha_3}) = p^{i+j-\alpha}(&C(p^{3\alpha_3-i},p^{3\alpha_3-j};p^{3\alpha_3}) - C(p^{3\alpha_3-i-1},p^{3\alpha_3-j};p^{3\alpha_3}) \\ &- C(p^{3\alpha_3-i},p^{3\alpha_3-j-1};p^{3\alpha_3}) + C(p^{3\alpha_3-i},p^{3\alpha_3-j-1};p^{3\alpha_3})). \end{aligned}$$

For all other i,j,α, $A(p^i,p^j;p^\alpha)=0$.

These formulas can be considerably simplified and we will find it particularly useful to do so in the case when α is large compared with $i+j$. We will refer to this as the *stable case*. The results of a straightforward computation applied to (2.6) and (2.7) are summarized in

Proposition 2.2. *For $0 \le i \le j$ and $\alpha \ge i+j$ we have, for $i+j \equiv 0, 2 \mod 3$,*

$$A(p^i, p^j; p^\alpha) = \begin{cases} p^{i+[(j-3)/2]} & \text{if } j \le 2i-1 \text{ and } i+j \equiv 2 \mod 3, \alpha \equiv 0 \mod 3; \\ p^{i+[(j)/2]} & \text{if } j \le 2i \text{ and } i+j \equiv 0 \mod 3. \end{cases}$$

For $i+j \equiv 1 \mod 3$ and $\alpha \equiv 0 \mod 3$,

$$A(p^{(i+j-1)/3}, p^{(2i+2j+1)/3}; p^\alpha) = -p^{2(i+j-1)/3}.$$

For $1+(i+j-1)/3 \le k_1 \le k_2 \le 2k_1+1$,

$$A(p^{k_1}, p^{k_2}; p^\alpha) = -p^{k_1+[(k_2-1)/2]} - p^{k_1+[(k_2-2)/2]}$$

unless $k_1 = k_2 = (i+j)/2$ and $i+j \equiv 10 \mod 12$, in which case

$$A(p^{(i+j)/2}, p^{(i+j)/2}; p^\alpha) = -2p^{3(i+j-2)/4}.$$

If $0 \le j \le i$, then $A(p^i, p^j; p^\alpha) = A(p^j, p^i; p^\alpha)$. For all other i, j, α with $\alpha \ge i+j$, $A(p^i, p^j; p^\alpha) = 0$.

Moreover, set

$$S(p^{k_1}, p^{k_2}; p^\alpha) = \sum_{0 \le c_1 \le k_1, 0 \le c_2 \le k_2} A(p^{c_1}, p^{c_2}; p^\alpha).$$

Then for $3\alpha_3 \ge k_1 + k_2$,

$$S(p^{k_1}, p^{k_2}; p^{3\alpha_3}) = A(p^{k_1}, p^{k_2}; p^{3\alpha_3 - 1}).$$

Fix $\psi^{(k)}, \psi^{(l)} \in \Psi^{(3)}$. We now define the fundamental multiple Dirichlet series

$$Z_1(s, w; \psi^{(k)}, \psi^{(l)}) = \sum_{d \equiv 1 \mod 3, (d,S)=1} \frac{L_S(s, \pi, \chi_{d_1 d_2^2} \psi^{(k)})\, P(s; d; \psi^{(k)})\, \psi^{(l)}(d)}{||d||^w}. \tag{2.9}$$

As usual we have written $d = d_1 d_2^2 d_3^3$, with each $d_i \equiv 1 \mod 3$, d_1, d_2 square-free and $(d_1, d_2) = 1$. The term $P(s; d; \psi^{(k)})$ is the key weight factor (or correction polynomial) and is defined by

$$P(s; d; \psi^{(k)}) = \prod_{p^{\alpha_3} || d_3} \sum_{0 \le i, j \le \alpha_2 + 3\alpha_3} A(p^i, p^j; p^{\alpha_2 + 3\alpha_3}) \times \chi_{d_1 d_2^2}(p^{(i+j)_0})\, \psi^{(k)}(p^{i+j})\, \gamma(p^i)\, \delta(p^j)\, ||p||^{-(i+j)s}. \tag{2.10}$$

Here the $A(p^i, p^j; p^{\alpha_2+3\alpha_3})$ are defined in (2.6)–(2.8). For $p^\alpha || d$ we have written $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3$, and we have also set $(i+j)_0 = 0, 1, 2$ so that $i + j \equiv (i+j)_0 \mod 3$.

Let $\overrightarrow{\mathbf{Z}}_1(s,w)$ denote the $C^2 \times 1$ vector whose entries are $Z_1(s, w; \psi^{(k)}, \psi^{(l)})$ and let

$$\overrightarrow{\mathbf{Z}}{}_1^*(s,w) = (w+2s-5/3)(w+2s-4/3)(w)(w-1)\zeta_S(6w+12s-8)\overrightarrow{\mathbf{Z}}_1(s,w).$$

We will now examine how $\overrightarrow{\mathbf{Z}}{}_1^*(s,w)$ behaves under the change of variables $(s,w) \to (1-s, w+2s-1)$. This and other properties of $\overrightarrow{\mathbf{Z}}{}_1^*(s,w)$ are summarized in

Proposition 2.3. *The function $\overrightarrow{\mathbf{Z}}{}_1^*(s,w)$ converges absolutely for $\Re s > 1, \Re w > 1$. It has analytic continuation to the region R_2 consisting of points (s,w) for which $\Re s > 1$ for $\Re w > 1$; $\Re w > 2 - \Re s$ for $1 \geq \Re s \geq 0$; and $\Re w > 2 - 2\Re s$ for $0 > \Re s$. Inside R_2 it satisfies the functional equations*

$$\overrightarrow{\mathbf{Z}}{}_1^*(s,w)\prod_{p|M}(1-\tilde{\alpha}_p^3||p||^{-3+3s})(1-\tilde{\beta}_p^3||p||^{-3+3s}) = \Phi_{1,6}(s)\overrightarrow{\mathbf{Z}}{}_6^*(1-s, w+2s-1).$$

and

$$\overrightarrow{\mathbf{Z}}{}_6^*(s,w)\prod_{p|M}(1-\alpha_p^3||p||^{-3+3s})(1-\beta_p^3||p||^{-3+3s}) = \Phi_{6,1}(s)\overrightarrow{\mathbf{Z}}{}_1^*(1-s, w+2s-1).$$

Here $\Phi_{1,6}(s), \Phi_{6,1}(s)$ are C^2 by ${C^{(6)}}^2$ matrices. Their entries are analytic functions in $\mathbb{C}$ except for possible poles corresponding to gamma factors in $\Gamma_{\pi,\psi}(s)$. These are canceled by trivial zeros of the L-series appearing in $\overrightarrow{\mathbf{L}}_S(1-s, \tilde{\pi}, \overline{\chi}_d)$. For s in bounded vertical strips, i.e. for $s = \sigma + it$ with $K_1 < \sigma < K_2$, the upper bounds $\Phi_{1,6}^{k,l}(s), \Phi_{6,1}^{k,l}(s) \ll |t|^A ||M||^B$ hold for $|t|$ sufficiently large and A, B depending on K_1, K_2. The implied constants depends only on the Casimir eigenvalues of π (that is, the eigenvalues with respect to the invariant differential operators).

Proof: For $\Re s > 1 + \epsilon$, $\epsilon > 0$, we have $L_S(s, \pi, \chi_{d_1 d_2^2}\psi^{(k)}) \ll 1$. From the description of R in (2.1)–(2.3) we have the trivial bound

$$R(p^a, p^b; p^c) \ll ||p||^{(a+b+c)/2}. \tag{2.11}$$

Similarly, from (2.6)–(2.8) together with Proposition 2.2 we have

$$C(p^{k_1}, p^{k_2}; p^\alpha) \ll ||p||^{\min(\alpha, k_1+k_2+\epsilon)}$$

and

$$A(p^i, p^j; p^\alpha) \ll ||p||^{i+j+\epsilon} \tag{2.12}$$

for any $\epsilon > 0$. This is sufficient to establish absolute convergence for $\Re s, \Re w > 1$.

Next consider the product

$$Z_1(s, w; \psi_1, \psi_2) \prod_{p|M} (1 - \tilde{\alpha}_p^3 ||p||^{-3+3s})(1 - \tilde{\beta}_p^3 ||p||^{-3+3s}).$$

Upon the change of variables $(s, w) \to (1 - s, w + 2s - 1)$, this reflects into a linear combination of functions of the form

$$\sum_{d \equiv 1 \mod 3, (d,S)=1} \frac{L_S(s, \tilde{\pi}, \overline{\chi}_{d_1 d_2^2} \psi_1') \, \overline{G^{(6)}(1, d_1)} \, G^{(6)}(1, d_2) \, P(1-s; d; \psi_1') \, \psi_2'(d) \, \tilde{\gamma}(d_2 d_3^3) \, \tilde{\delta}(d_2 d_3^3)}{||d||^w ||d_2 d_3^3||^{2s-1}}. \tag{2.13}$$

This follows from the functional equation of Proposition 1.1; the 6th order Gauss sums arise here after factoring the square of a cubic Gauss sum and applying the Hasse-Davenport relation [DH]. Specifically, we note that

$$\tau(\chi_{d_1 d_2^2})^2 = G^{(3)}(1, d_1)^2 \, \overline{G^{(3)}(1, d_2)}^2$$

and by the Hasse-Davenport relation

$$\left(\frac{2}{d_1}\right)_3^{-1} G^{(3)}(1, d_1)^2 = \overline{G^{(6)}(1, d_1)}$$

and

$$\left(\frac{2}{d_2}\right)_3 \overline{G^{(3)}(1, d_2)}^2 = G^{(6)}(1, d_2).$$

The residue symbols here are then absorbed into the ψ_2' above.

We next observe that if the expression in (2.13) is expanded and the order of summation interchanged, one obtains exactly $Z_6(s, w; \psi_1', \psi_2')$. This is not surprising, as the definitions of $A(p^i, p^j; p^\alpha)$ given in (2.6)–(2.8) were made in order that this identity should hold. More precisely, if one applies an interchange in the order of summation to the definition of $Z_6(s, w; \psi_1, \psi_2)$ in (2.4) and then uses (2.13) as a definition of the functions $A(p^i,p^j;p^\alpha)$, one recovers the functions described in (2.6)–(2.8). The functional equations of the Proposition follow from this relation.

The analytic continuation of $\overrightarrow{\mathbf{Z}}{}_1^*(s,w)$ for $\Re s<0$ now follows by combining (2.13), Proposition 2.1, and the bounds on the coefficients of the linear combination following from Proposition 1.1. For $0\le\Re s\le 1$ the continuation then follows from the Phragmén-Lindelöf principle.

Finally, we must deal with the possibility of hidden poles of $\overrightarrow{\mathbf{Z}}{}_1^*(s,w)$ cancelled by zeros of $\prod_{p|M}(1-\tilde{\alpha}_p^3||p||^{-3+3s})(1-\tilde{\beta}_p^3||p||^{-3+3s})$. Such poles would then continue into polar lines for fixed s and large values of w. These are ruled out by the analytic behavior of $L_S(s,\pi,\chi_{d_1d_2^2}\psi^{(k)})$ for all s and the absolute convergence of $\overrightarrow{\mathbf{Z}}{}_1^*(s,w)$ for any fixed s when the real part of w is sufficiently large. This completes the proof of Proposition 2.3. ■

Remark 2.4. The correction polynomials which are specified above via the coefficients $R(p^a,p^b;p^c)$, $A(p^i,p^j;p^\alpha)$ are crucial to making these arguments work. Indeed, the basic method—combining functional equations with interchange of summation in the double Dirichlet series—must take into account that the variables in the double summation need not be square-free and relatively prime. Moreover, the functional equations given by Propositions 2.1 and 2.3 must remain valid even with the correction polynomials added. Observe that the class of functions

$$\sum_m \frac{\sum_{r|m}F(m,r,w)D(w,m^2/r^2)}{||m||^s}$$

where the $F(m,r,w)$ are finite, Eulerian, Dirichlet polynomials such that $F(m,r,1-w)=||r||^{1-2w}F(m,r,w)$ all satisfy the functional equation $(s,w)\to(s+2w-1,1-w)$ of Proposition 2.1. If in addition one requires the $F(m,r,w)$ to define corresponding Dirichlet polynomials in $Z_1(s,w)$, according to the functional equation $Z_6(s,w)\to Z_1(s+2w-1,1-w)$, with a list of desired properties [Br], one obtains the polynomials given above. Surprisingly, the $F(m,r,w)$ turn out to be monomials of the form $F(m,r)||r||^{-w}$.

The stability properties for $A(p^i,p^j;p^\alpha)$ described in Proposition 2.2 allow an interchange in the order of summation of $Z_1(s,w;\psi^{(k)},\psi^{(l)})$ which expresses this series in terms of Hecke L-series. To state this, we define the weight factors written $Q(w;m)$ in the introduction. To give these (in somewhat greater generality, as we include the characters $\psi^{(\ell)}$), let $m,n\equiv 1 \mod 3$ with $(mn,S)=1$ and write $mn=(mn)_0(mn)_3^3$, where $(mn)_0$ is the cube free part of mn. Also, if p a prime with $p^{k_1}||m$ and $p^{k_2}||n$ write $k_1+k_2=\beta_0+3\beta_3$ with $\beta_0=\beta_1+2\beta_2=0,1,2$ and for $r\le\beta_2+3\beta_3$ similarly write $r=r_0+3r_3$. Let $\psi^{(\ell)}\in\Psi^{(3)}$. Then we define

$$Q(w;m,n;\psi^{(l)})=\prod_{p^{\beta_3}||(mn)_3}\sum_{0\le r\le\beta_2+3\beta_3}B(p^r;p^{k_1},p^{k_2})$$
$$\times\chi_{(mn)_0}(p^{r_0})\,\psi^{(l)}(p^{k_1+k_2})\,\gamma(p^{k_1})\,\delta(p^{k_2})\,||p||^{-rw}.$$

For $\beta_0=0$ and $r_3\ge1$ the functions $B(p^r;p^{k_1},p^{k_2})$ are given by

$$B(p^{3r_3};p^{k_1},p^{k_2})=S(p^{k_1},p^{k_2};p^{3r_3})-A(p^{k_1},p^{k_2};p^{3r_3-1})$$

$$B(p^{3r_3-1};p^{k_1},p^{k_2})=A(p^{k_1},p^{k_2};p^{3r_3-1})-A(p^{k_1},p^{k_2};p^{3r_3-2})$$
$$B(p^{3r_3-2};p^{k_1},p^{k_2})=A(p^{k_1},p^{k_2};p^{3r_3-2})-S(p^{k_1},p^{k_2};p^{3r_3-3}).$$

For $\beta_0=1,2$, $B(p^{r_0+3r_3};p^{k_1},p^{k_2})=0$ when $r_0=1,2$. For $r_0=0$

$$B(p^{3r_3};p^{k_1},p^{k_2})=S(p^{k_1},p^{k_2};p^{3r_3}).$$

For $k_1,k_2\ge0$

$$B(1;p^{k_1},p^{k_2})=1 \tag{2.14}$$

while for $(k_1,k_2)=(2,1),(1,2),(2,2)$

$$B(p^3;p^{k_1},p^{k_2})=||p||^2 \tag{2.15}$$

and $B(p^{\beta_2+3\beta_3};p^{k_1},p^{k_2})=0$ for all other $(k_1,k_2)\ne(0,0)$. Then we have

Proposition 2.5. *The function $Z_1(s,w;\psi^{(k)},\psi^{(l)})$ defined in (2.9) can be expressed in the form*

$$Z_1(s,w;\psi^{(k)},\psi^{(l)})=$$
$$\sum_{m,n\equiv1\mod 3,\,(mn,S)=1}\frac{L_S(w,\chi_{(mn)_0}\psi^{(l)})\,Q(w;m,n;\psi^{(l)})\,\psi^{(k)}(mn)}{||mn||^s}.$$

This series is absolutely convergent for fixed w and $\Re s$ sufficiently large. It extends to an analytic function of (s,w) in the region R_3 defined by $\Re w>2-2\Re s$ for

$\Re s < 0$; $\Re w > 2 - (4/3)\Re s$ for $0 \leq \Re s \leq 3/2$; $\Re w > 3/2 - \Re s$ for $\Re s > 3/2$. Finally, the function $B(p^r; p^{k_1}, p^{k_2})$ satisfies the bound

$$B(p^r; p^{k_1}, p^{k_2}) \ll ||p||^{k_1+k_2+\epsilon}$$

for any $\epsilon > 0$.

Proof: Equating the expression for $Z_1(s, w; \psi^{(k)}, \psi^{(l)})$ given in (2.9), (2.10) with that in the Proposition and comparing coefficients of $\gamma(p^{k_1})\delta(p^{k_2})$ we obtain the basic required relation

$$\sum_{0 \leq r \leq \beta_2+3\beta_3} \frac{B(p^r; p^{k_1}, p^{k_2})}{p^{(\beta_2+3\beta_3-r)(w-1/2)}} = \sum_{c_1+l_1=k_1, c_2+l_2=k_2} ||p||^{-(\beta_2+3\beta_3)(w-1/2)}$$

$$\times \left(\sum_{\alpha \geq 0} \frac{\chi_{p^{\alpha_0}}(p^{l_1+l_2})\, A(p^{c_1}, p^{c_2}; p^\alpha)}{p^{\alpha(1-w)}} - \chi_{p^{\beta_0}}(p) \sum_{\alpha \geq 0} \frac{\chi_{p^{\alpha_0}}(p^{l_1+l_2})\, A(p^{c_1}, p^{c_2}; p^\alpha)}{p^{(\alpha+1)(1-w)}} \right).$$

The relation itself follows from a comparison of the coefficients of $||p||^{-Kw}$ for $K \geq 0$. Note that the stability properties of Proposition 2.2 are equivalent to the relation $B(p^r; p^{k_1}, p^{k_2}) = 0$ for $r > \beta_2 + 3\beta_3$.

The upper bound for $B(p^r; p^{k_1}, p^{k_2})$ follows from these expressions and the bound (2.12). To verify continuation to the region R_3 consider first the subsum over $(mn)_3$ while holding $(mn)_0$ fixed:

$$Z_1(s, w; \psi^{(k)}, \psi^{(l)}; (mn)_0) = \sum_{(mn)_3} \frac{Q(w; m, n; \psi^{(l)})\, \psi^{(k)}(mn)}{||(mn)_3||^{3s}}.$$

For $\Re w < 0$, the upper bound for $B(p^r; p^{k_1}, p^{k_2})$ implies that

$$Q(w; m, n; \psi^{(l)}) \ll ||mn||^{1-\Re w+\epsilon}.$$

This, together with the well known absolute convergence of $\sum |c(m)|^4 ||m||^{-u}$ for $\Re u > 1$, implies the absolute convergence of $Z_1(s, w; \psi^{(k)}, \psi^{(l)}; (mn)_0)$ for $\Re w < 0$ and $\Re s + w > 4/3$. For $0 \leq \Re w \leq 1$ we have absolute convergence for $\Re s > 4/3$. The usual convexity bounds for $L(w, \chi_{(mn)_0}\psi^{(l)})$ together with the convergence of $\sum |c(m)|^2 ||m||^{-u}$ for $\Re u > 1$ then imply the convergence of the sum over $(mn)_0$ in the regions $\Re w < 0, \Re s > 3/2 - \Re w$ and $0 \leq \Re w \leq 1$ with $\Re s > 3/2$. Interchanging the order of summation and recalling the region of convergence R_2 given in Proposition 2.3 we obtain the analytic continuation of $Z_1(s, w; \psi^{(k)}, \psi^{(l)})$ to R_3, which is the convex hull of the union of these regions. ■

3. The analytic continuation. The first objective of this section is to establish the following result.

Proposition 3.1. *The functions* $\overrightarrow{\mathbf{Z}}^*_1(s,w)$ *and* $\overrightarrow{\mathbf{Z}}^*_6(s,w)$ *have analytic continuation to the half plane* $\Re(w+s) > 3/2$.

Proof: In the previous section we established that $\overrightarrow{\mathbf{Z}}^*_6(s,w)$ is analytic in R_1 (Proposition 2.1) and that $\overrightarrow{\mathbf{Z}}^*_1(s,w)$ is analytic in R_3 (Proposition 2.5). Applying the second functional equation of Proposition 2.3 and the convexity principle we see that $\overrightarrow{\mathbf{Z}}^*_6(s,w)$ is analytic in the convex hull of the union of R_1 and the image of R_3 under the transformation $(s,w) \to (1-s, w+2s-1)$. This is the region R_4 defined by: $\Re w > -\Re s + 3/2$ for $\Re s \leq -1/2$, $\Re w > -(4/5)\Re s + 8/5$ for $-1/2 \leq \Re s \leq 2$, $\Re w > -(1/2)\Re s + 1$ for $\Re s > 2$. The possibility of poles hidden by the product $\prod_{p|M}(1-\alpha_p^3 ||p||^{-3+3s})(1-\beta_p^3||p||^{-3+3s})$ is eliminated by taking w to have large real part.

We now apply Proposition 2.1 reflecting $\overrightarrow{\mathbf{Z}}^*_6(s,w)$ into itself. This establishes the analytic continuation of $\overrightarrow{\mathbf{Z}}^*_6(s,w)$ into the region R_5 defined by the convex hull of the union of R_4 with its image under the transformation $(s,w) \to (s+2w-1, 1-w)$. This is the half plane $\Re(w+s) > 3/2$. There are possible poles concealed at $\Re w = 2/3$, but except for the point $w = 2/3$ these are ruled out by the absolute convergence of (2.4) for any fixed w and sufficiently large $\Re s$.

Finally, applying the first functional equation of Proposition 2.3 we obtain the analytic continuation of $\overrightarrow{\mathbf{Z}}^*_1(s,w)$ to the half plane $\Re(w+s) > 3/2$. Once again possible poles of $\overrightarrow{\mathbf{Z}}^*_1(s,w)$ at fixed values of s concealed by the product $\prod_{p|M}(1-\tilde{\alpha}_p^3||p||^{-3+3s})(1-\tilde{\beta}_p^3||p||^{-3+3s})$ can be ruled out by the analyticity of the function $L_S(s,\pi,\chi_{d_1 d_2^2}\psi^{(k)})$ for all s and the absolute convergence of $\overrightarrow{\mathbf{Z}}^*_1(s,w)$ for any fixed s when the real part of w is sufficiently large. This completes the proof of Proposition 3.1. ■

We will now extend the analytic continuation of $\overrightarrow{\mathbf{Z}}^*_1(s,w)$ to the half plane $\Re(w+s) > 1/2$. This will take some work; the result is stated in Proposition 3.3 below. To accomplish this we define a third multiple Dirichlet series. Recall the notation $mn = (mn)_0(mn)_3^3$, where $(mn)_0$ is the cube free-part of mn. Further refining this, we write $(mn)_0 = (mn)_1 {(mn)_2}^2$ with $(mn)_1$ square-free. Let

$$Z_3(s,w;\psi_1,\psi_2) = \sum_{m,n\equiv 1 \mod 3,\,(mn,S)=1} \frac{L_S(w,\overline{\chi}_{(mn)_0}\overline{\psi_2})\, G(1,(mn)_1)\, \overline{G(1,(mn)_2)}\, Q(1-w;m,n;\psi_2)\, \psi_1(mn)}{||mn||^s ||(mn)_2 {(mn)_3}^3||^{w-1/2}}.$$

By expanding $Q(1-w;m,n;\psi_2)$ using the definition of these functions above Proposition 2.5, one obtains an expression for $Z_3(s,w;\psi_1,\psi_2)$ which is easier to work with, namely

$$(3.1)\quad Z_3(s,w;\psi_1,\psi_2) = \sum_{m,n,R} L_S(w,\overline{\chi}_{(mn)_0}\overline{\psi_2})\, G(1,(mn)_1)\, \overline{G(1,(mn)_2)} \times \frac{B(R;m,n)\, \psi_1(mn)\, \psi_2(R)\, \chi_{(mn)_0}(R_0)\, \gamma(m)\, \delta(n)}{||mn||^s ||(mn)_2 {(mn)_3}^3||^{w-1/2} ||R||^{1-w}}.$$

Here the sum is over $m,n,R \equiv 1 \mod 3$ with $(mnR,S)=1$ and $R|(mn)_2 {(mn)_3}^3$. We have also extended the definition of $B(p^r;,p^{k_1},p^{k_2})$ multiplicatively to all $B(R;m,n)$ by

$$B(RR';mm',nn') = B(R;m,n)\, B(R';m',n')$$

for $(Rmn,R'm'n')=1$.

It is clear from the expression (3.1) that $Z_3(s,w;\psi_1,\psi_2)$ converges absolutely for fixed w and s with sufficiently large real parts. Let $\overrightarrow{\mathbf{Z}}_3(s,w)$ denote the $C^2 \times 1$ vector whose entries are the functions $Z_3(s,w;\psi^{(k)},\psi^{(l)})$ and let

$$\overrightarrow{\mathbf{Z}}^*_3(s,w) = (w+2s-5/3)(w+2s-4/3)(w)(w-1)\zeta_S(6w+12s-8)\,\overrightarrow{\mathbf{Z}}_3(s,w).$$

Then it is straightforward to obtain

Proposition 3.2. *The function* $\overrightarrow{\mathbf{Z}}^*_3(s,w)$ *converges absolutely for* $\Re s > 1, \Re w > 1$. *It also has an analytic continuation to the half plane* $\Re s > 1$. *Inside this half plane it satisfies the functional equations*

$$\overrightarrow{\mathbf{Z}}^*_1(s,w) \prod_{p|M} (1 - ||p||^{-3+3w}) = \Phi_{1,3}(w)\ \overrightarrow{\mathbf{Z}}^*_3(s+w-1/2, 1-w).$$

and

$$\overrightarrow{\mathbf{Z}}^{*}_{3}(s,w)\prod_{p|M}(1-||p||^{-3+3w}) = \Phi_{3,1}(w)\;\overrightarrow{\mathbf{Z}}^{*}_{1}(s+w-1/2,1-w).$$

Here $\Phi_{1,3}(w), \Phi_{3,1}(w)$ *are* C^2 *by* $(C^{(6)})^2$ *matrices. Their entries are analytic functions in* $\mathbb{C}$ *except for possible poles corresponding to gamma factors in* $\Gamma_\psi(w)$*. These are canceled by trivial zeros of the L-series appearing in the numerator. For* w *in bounded vertical strips, i.e. for* $w = \sigma + it$ *with* $K_1 < \sigma < K_2$*, the upper bound* $\Phi^{k,l}_{1,3}(s), \Phi^{k,l}_{3,1}(w) \ll |t|^A||M||^B$ *holds for* $|t|$ *sufficiently large and* A, B *depending on* K_1, K_2*. The implied constant depends only on* M*,* K *and the Casimir eigenvalues of* π*.*

Proof: The functional equations follow immediately from (3.1) and Proposition 1.2. The analytic continuation follows as the half plane $\Re s > 1$ is the image of the half plane $\Re(s+w) > 3/2$ under the transformation $(s,w) \to (s+w-1/2, 1-w)$. Finally, the product $(w+2s-5/3)(w+2s-4/3)(w)(w-1)\zeta_S(6w+12s-8)$ is fixed under this transformation. ■

The next step is to reverse the order of summation in (3.1) and to find an alternative expression for $Z_3(s,w;\psi_1,\psi_2)$ as a sum of twisted Rankin-Selberg convolutions. Expanding the L-series in the numerator we obtain, for $\Re w, \Re s > 1$,

$$Z_3(s,w;\psi_1,\psi_2) = \sum_{m,n,R,d} \overline{\chi}_{(mn)_0}(d)\,\overline{\psi_2}(d)\,G(1,(mn)_1)\,\overline{G(1,(mn)_2)}$$
$$\times\;\frac{B(R;m,n)\,\psi_1(mn)\,\psi_2(R)\,\chi_{(mn)_0}(R_0)\,\gamma(m)\,\delta(n)}{||mn||^{s}\,||(mn)_2(mn)_3{}^3/R||^{w-1/2}\,||R||^{1/2}}.$$

with the sum over $m,n,R,d \equiv 1 \mod 3$, $(mnR,S) = 1$ and $R|(mn)_2(mn)_3{}^3$. Let us now make a change of variables, setting $\mathcal{M} = mn = \mathcal{M}_1\mathcal{M}_2^2\mathcal{M}_3^3$ and $\mathcal{D} = d\mathcal{M}_2\mathcal{M}_3^3/R$. Here, as usual, $M_1M_2^2$ is the cube-free part of M and M_1, M_2 are square-free with $(M_1,M_2) = 1$. With this notation the sum becomes

$$Z_3(s,w;\psi_1,\psi_2) = \sum_{\mathcal{M},R,\mathcal{D}} \overline{\chi}_{\mathcal{M}_1\mathcal{M}_2^2}(\mathcal{D}R/(\mathcal{M}_2\mathcal{M}_3^3))\,\overline{\psi_2}(\mathcal{D}R/\mathcal{M}_2)$$
$$\times\;\frac{G(1,\mathcal{M}_1)\,\overline{G(1,\mathcal{M}_2)}\,\psi_1(\mathcal{M})\,\psi_2(R)\,\chi_{\mathcal{M}_1\mathcal{M}_2^2}(R_0)\,\sum_{\mathcal{M}=mn}B(R;m,n)\gamma(m)\delta(n)}{||\mathcal{D}||^{w}\,||\mathcal{M}||^{s}\,||\mathcal{M}_2\mathcal{M}_3^3/R||^{-1/2}\,||R||^{1/2}}.$$

The sum is over $\mathcal{M}, R, \mathcal{D} \equiv 1 \mod 3$, $(\mathcal{M}R, S) = 1$, with the requirements that $R|\mathcal{M}_2\mathcal{M}^3$ and $(\mathcal{M}_2\mathcal{M}_3^3/R)|D$.

It will be convenient to write $\mathcal{M} = \mathcal{M}_{\mathcal{D}}\mathcal{M}'$, with $\mathcal{M}_{\mathcal{D}}|\mathcal{D}^\infty$ and $(\mathcal{M}', \mathcal{D}) = 1$. We then have corresponding factorizations $\mathcal{M} = \mathcal{M}_{\mathcal{D},1}\mathcal{M}_{\mathcal{D},2}^2\mathcal{M}_{\mathcal{D},3}^3\mathcal{M}_1'{\mathcal{M}_2'}^2{\mathcal{M}_3'}^3$ and $R = R_{\mathcal{D}}R'$. As $(\mathcal{M}_2\mathcal{M}_3^3/R)|D$ it follows that $(\mathcal{M}_{\mathcal{D},2}\mathcal{M}_{\mathcal{D},3}^3/R_{\mathcal{D}})|\mathcal{D}$ and $R' = \mathcal{M}_2'{\mathcal{M}_3'}^3$. In the above sum

$$\chi_{\mathcal{M}_1\mathcal{M}_2^2}(R_0) = \chi_{\mathcal{M}_1\mathcal{M}_2^2}(R_{\mathcal{D},0})\chi_{\mathcal{M}_1\mathcal{M}_2^2}(\mathcal{M}_2')$$

and so there will be no contribution unless $\mathcal{M}_2' = 1$. Taking this into account and factoring the Gauss sums we obtain

$$\begin{aligned}(3.2)\quad & Z_3(s, w; \psi_1, \psi_2) = \\ & \sum_{\mathcal{D}, \mathcal{M}_{\mathcal{D}}|D^\infty} \overline{\chi}_{\mathcal{M}_{\mathcal{D},1}\mathcal{M}_{\mathcal{D},2}^2}(\mathcal{D}R_{\mathcal{D}}R_{\mathcal{D},0}^2/(\mathcal{M}_{\mathcal{D},2}\mathcal{M}_{\mathcal{D},3}^3))\,\overline{\psi_2}(\mathcal{D}/\mathcal{M}_{\mathcal{D},2})\,G(1, \mathcal{M}_{\mathcal{D},1}) \\ & \times \overline{G(1, \mathcal{M}_{\mathcal{D},2})}\frac{\psi_1(\mathcal{M}_{\mathcal{D}})\sum_{\mathcal{M}_{\mathcal{D}}=mn} B(R_{\mathcal{D}}; m, n)\,\gamma(m)\,\delta(n)\,Z_3(s, \mathcal{D}, \mathcal{M}_{\mathcal{D}}; \psi_1)}{||\mathcal{D}||^w||\mathcal{M}_{\mathcal{D}}||^s||\mathcal{M}_{\mathcal{D},2}\mathcal{M}_{\mathcal{D},3}^3/R_{\mathcal{D}}||^{-1/2}||R_{\mathcal{D}}||^{1/2}}.\end{aligned}$$

where

$$\begin{aligned}& Z_3(s, \mathcal{D}, \mathcal{M}_{\mathcal{D}}; \psi_1) = \\ & \sum_{\mathcal{M}_1, \mathcal{M}_3} \frac{\overline{\chi}_{\mathcal{M}_1'}(\mathcal{D}M_{\mathcal{D},2}^2M_{\mathcal{D},1})\,G(1, \mathcal{M}_1')\,\psi_1(\mathcal{M}')\sum_{\mathcal{M}'=mn} B({\mathcal{M}_3'}^3; m, n)\,\gamma(m)\,\delta(n)}{||\mathcal{M}'||^s||\mathcal{M}_3'||^{3/2}}.\end{aligned}$$

Fortunately, $B({\mathcal{M}_3'}^3; m, n)$ has a very simple description, given in (2.14), (2.15). It follows from this description that $Z_3(s, \mathcal{D}, \mathcal{M}_{\mathcal{D}}; \psi_1)$ is essentially the cube-free part of the Dirichlet series which represents the Rankin-Selberg convolution $L_S(s, \pi \otimes \overline{\theta^{(3)}} \otimes \overline{\chi}_d\psi_{\mathcal{D}})$ for some $\psi_{\mathcal{D}}$ depending on ψ_1 and $\mathcal{D}$. More precisely, for such a $\psi_{\mathcal{D}}$ one has

$$\begin{aligned}(3.3)\quad & \prod_{p \notin S}(1 - \gamma_p^3||p||^{1/2-3s})^{-1}(1 - \delta_p^3||p||^{1/2-3s})^{-1}Z_3(s, \mathcal{D}, \mathcal{M}_{\mathcal{D}}; \psi_1) \\ & \qquad = L_S(s, \pi \otimes \overline{\theta^{(3)}} \otimes \overline{\chi}_d\psi_{\mathcal{D}}).\end{aligned}$$

It follows from (3.2), (3.3) and Proposition 1.4 that there exists a $\kappa > 0$ such that for $\Re s > 0$ and $\Re w > \kappa$,

$$\mathcal{P}(s)(w + 2s - 5/3)(w + 2s - 4/3)(w)(w - 1)\zeta_S(6w + 12s - 8)Z_3(s, w; \psi_1, \psi_2)$$

is an analytic function of s and w and converges absolutely, where

$$\mathcal{P}(s) = (s-1/3)(s-2/3)\zeta_S(6s-2)\prod_{p\notin S}(1-\gamma_p^3||p||^{1/2-3s})^{-1}(1-\delta_p^3||p||^{1/2-3s})^{-1}.$$

Thus for $\Re s > 0$ and $\Re w > \kappa$, $\mathcal{P}(s)\overrightarrow{\mathbf{Z}}{}_3^*(s,w)$ is an analytic function of s and w and converges absolutely. Now by Proposition 3.2, $\mathcal{P}(s)\overrightarrow{\mathbf{Z}}{}_3^*(s,w)$ is also analytic in the half plane $\Re s > 1$. Taking the convex hull of these two overlapping regions we see that $\mathcal{P}(s)\overrightarrow{\mathbf{Z}}{}_3^*(s,w)$ is analytic in the half plane $\Re s > 0$. Applying the functional equation of Proposition 3.2 we obtain the desired analytic continuation:

Proposition 3.3. *The function*

$$\overrightarrow{\mathbf{Z}}{}_1^*(s,w)(s+w-7/6)(s+w-5/6)\zeta_S(6s+6w-5) \\ \times\prod_{p\notin S}(1-\gamma_p^3||p||^{1/2-3s})^{-1}(1-\delta_p^3||p||^{1/2-3s})^{-1}$$

has analytic continuation to the half plane $\Re(w+s) > 1/2$.

4. Proof of Theorem 0.1 and the symmetric cube as a residue. We first establish

Proposition 4.1. *Let* $\Re(s) > -1/2$. *Then*

$$\mathrm{Res}_{w=1} Z_1(s,w;1,1) = \\ c_S\, L_S(3s,\pi,\vee^3)\,\zeta_S(6s)\,\zeta_S(6s+1)^{-1}\prod_{p\notin S}(1-\gamma_p^3||p||^{-3s-1})(1-\delta_p^3||p||^{-3s-1}),$$

where c_S *is the non-zero constant given in (4.2) below.*

Proof: Consider $Z_1(s,w;\psi^{(k)},\psi^{(l)})$ given in Proposition 2.5. For $\Re s > 1$ the sum converges absolutely. Taking $\psi^{(k)} = \psi^{(l)} = 1$ there is clearly a simple pole at $w=1$ which picks off the terms in the summation corresponding to $(mn,S)=1$ and mn a cube. Taking the residue at $w=1$ we obtain

$$\mathrm{Res}_{w=1} Z_1(s,w;1,1) = c_S \sum_{(mn,S)=1,(mn)_0=1}\frac{Q(1;m,n;1)}{||mn||^s}, \tag{4.1}$$

where

$$c_S = \prod_{p|M} \left(1 - ||p||^{-1}\right) \tag{4.2}$$

is a non-zero constant. To complete the proof it suffices to show that right hand side of (4.1) is the desired multiple of $L_S(3s, \pi, \vee^3)$; the result for $\Re(s) > -1/2$ then follows from Proposition 3.3. We must show that

$$\begin{aligned}\zeta_S(6s+1) \sum_{(mn,S)=1,(mn)_0=1} \frac{Q(1;m,n;1)}{||mn||^s} \\ = L_S(3s,\pi,\vee^3)\zeta_S(6s) \prod_{p\notin S} (1-\gamma_p^3||p||^{-3s-1})(1-\delta_p^3||p||^{-3s-1}).\end{aligned}$$

This is proved by a local calculation. As

$$\sum_{(mn,S)=1,(mn)_0=1} \frac{Q(1;m,n;1)}{||mn||^s} = \prod_{p\notin S}\left(1+\sum_{k=1}^{\infty} \frac{\sum_{k_1+k_2=3k} Q(1,p^{k_1},p^{k_2};1)}{||p||^{3ks}}\right)$$

we must verify that for $p \notin S$

$$\begin{aligned}1+\sum_{k=1}^{\infty} \frac{\sum_{k_1+k_2=3k} Q(1,p^{k_1},p^{k_2};1)}{||p||^{3ks}} &= (1-||p||^{-1-6s})\,(1-||p||^{-6s})^{-1} \\ &\times (1-\gamma_p^3||p||^{-3s-1})\,(1-\delta_p^3||p||^{-3s-1})\,L_p(3s,\pi,\vee^3)^{-1},\end{aligned} \tag{4.3}$$

where

$$L_p(3s,\pi,\vee^3) = (1-\gamma_p^3||p||^{-3s})(1-\gamma_p^2\delta_p||p||^{-3s})(1-\gamma_p\delta_p^2||p||^{-3s})(1-\delta_p^3||p||^{-3s})$$

is the local Euler factor of the symmetric cube L-function. By Proposition 2.5, for each $k_1, k_2 \geq 0, k_1+k_2 = 3k$ we have

$$Q(1,p^{k_1},p^{k_2};1) = \sum_{0\leq r\leq 3k} B(p^r;p^{k_1},p^{k_2})\gamma(p^{k_1})\delta(p^{k_2})||p||^{-r}.$$

The values of $B(p^r;p^{k_1},p^{k_2})$ are given in Proposition 2.5 and the identity (4.3) follows from a long but straightforward calculation. To give the flavor of it, we will check equality of the coefficients of $||p||^{-3s}$.

Using the evaluations of $B(p^r; p^{k_1}, p^{k_2})$ for $k_1 + k_2 = 3$ and $0 \le r \le 3$ we obtain

$$\begin{aligned} &1 + B(p; p^{k_1}, p^{k_2})||p||^{-1} + B(p^2; p^{k_1}, p^{k_2})||p||^{-2} + B(p^3; p^{k_1}, p^{k_2})||p||^{-3} \\ &\qquad = \begin{cases} 1 - ||p||^{-1} + 0 + ||p||^{2-3} = 1 & \text{for } (k_1, k_2) = (2,1), (1,2); \\ 1 - ||p||^{-1} & \text{for } (k_1, k_2) = (3,0), (0,3). \end{cases} \end{aligned}$$

It follows that

$$\sum_{k_1+k_2=3} Q(1, p^{k_1}, p^{k_2}; 1) = \gamma_p^2 \delta_p + \gamma_p \delta_p^2 + \gamma_p^3 - \gamma_p^3 ||p||^{-1} + \delta_p^3 - \delta_p^3 ||p||^{-1},$$

which is in agreement with the right hand side of (4.3). ■

Proof of Theorem 0.1 and Corollary 0.2: Observe that $\overrightarrow{\mathbf{Z}}_1(s, w)$ and hence $Z_1(s, w; 1, 1)$ has been shown to have polynomial growth in $\Im w$ for fixed s. Let us compute

$$I(s, X) = \frac{k!}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{Z_1(s, w; 1, 1) X^w dw}{w(w+1)\cdots(w+k)}.$$

Evaluating this term by term gives the left hand side of the Theorem. On the other hand, we may move the line of integration slightly past $\Re w = 1$, picking off the residue at $w = 1$ plus an error term. Because of the polynomial growth the integral converges and the error term is $\mathrm{O}(X^{1-\epsilon})$ for $\epsilon > 0$, as long as k is sufficiently large. The evaluation of the residue gives the right hand side of the Theorem. Of course a more refined analysis is possible and one could reduce the error term and reveal further main terms. To prove Corollary 0.2 it suffices to note that the convexity bound $L_S(1/2, \pi, \chi^{(3)}_{d_1 d_2^2}) P(1/2; d) \ll ||d||^{1/2}$ holds. Each non-zero term can be repeated with multiplicity at most $X^{1/3}$, giving the result. ■

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