Math 109, Winter 04
Theorem on Conjugacy Classes of S_n

Recall that in class we showed that conjugation is an equivalence relation. That is, if we define the relation $\sim$ on a group G according to $x \sim y$ if

$$\text{there exists a } g \in G \text{ such that} g * x * g^{-1} = y$$

then $\sim$ is an equivalence relation. So the equivalence classes under conjugation form a partition of the set G according to another result from class. We are excited because without showing that conjugation is an equivalence relation, it would be hard to show that the classes partition G. By brute force (that is, performing all the conjugations) we can show that there are three conjugacy classes in S_3. They are

$$\{e\}, \quad \{(1\ 2), (2\ 3), (1\ 3)\} \quad \{(1\ 2\ 3), (1\ 3\ 2)\}$$

In fact, a similar pattern is true in general according to the following result.

Theorem 1. *For any positive integer n, conjugacy classes of S_n are given by the collection of elements of the same cycle type.*

Proof: We must show two statements. First, that conjugation by an element g preserves cycle structure. And second, that all possible combinations of a given cycle type are in the SAME conjugacy class.

We start with the first statement. Given a permutation P, we know that we may write P as a disjoint product of cycles, say $C_1, C_2, \ldots, C_k$. We must show that gPg^{-1} has k cycles of the same size. But

$$gPg^{-1} = gC_1C_2 \cdots C_k g^{-1} = gC_1g^{-1}gC_2g^{-1} \cdots gC_kg^{-1}.$$

So if we can just show that for any cycle C of length m and any $g \in S_n$, gCg^{-1} is also a cycle of the same length m, then by the above equality, we have shown the first statement.

Write our cycle of length m as

$$C = (a_1\ a_2\ \cdots\ a_m).$$

Then for any $g \in S_n$, we will write $g(a_1)$ for the image of a_1 under g. Similarly g takes a_2 to $g(a_2)$ and in general, g takes a_i to $g(a_i)$. We claim that

$$gCg^{-1} = (g(a_1)\ g(a_2)\ \cdots\ g(a_m))$$

a cycle of length m. One can check that gCg^{-1} takes $g(a_i)$ to $g(a_{i+1})$ since g^{-1} takes $g(a_i)$ to a_i, then C takes a_i to a_{i+1}, then g takes a_{i+1} to $g(a_{i+1})$.

If there is a number not in this list $g(a_1), \ldots, g(a_k)$, call it r, we claim gCg^{-1} fixes r. This follows because C only moves elements of the form $a_i = g^{-1}(g(a_i))$ and r is not one of the $g(a_i)$, so $g^{-1}(r)$ is not one of the $g^{-1}(g(a_i))$. Hence C fixes $g^{-1}(r)$. Applying g to $g^{-1}(r)$ gives r. So gCg^{-1} really is a cycle of length m given by $(g(a_1)\ g(a_2)\ \cdots\ g(a_m))$. This completes the proof of the first statement.

To prove the second statement, that every permutation of a given cycle type is in the same conjugacy class. we must find a g such that if P_1 and P_2 are any two permutations of the same type, then $gP_1g^{-1} = P_2$. Let's do a motivating example first. Suppose P_1 is a single cycle (1 2 3 4) and P_2 is (6 7 8 9). If g takes 1 to 6, 2 to 7, 3 to 8 and 4 to 9, then one can check the equality holds. (Think about why this makes sense.) In general, given two permutations P_1, P_2 of the same cycle type, let g map elements of any cycle in P_1 to the corresponding elements in a cycle of the same size in P_2. Numbers not included in any cycle of P_1 can be taken to any number not included in cycles of P_2. Then we have taken each number 1 through n to a unique number 1 through n, so g is a bona fide permutation. Moreover, $gP_1g^{-1} = P_2$ by definition, as was shown in the example.